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Article

Analysis of Required Quantization Precision for CIE XYZ Signals in Discrete Region

Isao FURUKAWA* and Junji SUZUKI**

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Abstract

The Digital Cinema Initiative (DCI) in the U.S. has standardized a system for digital cinema in 2005, which digitally processes movies from production to distribution/screening. In the DCI standard, the required number of quantization bits is determined by applying the Barten's model as a visual model for luminance signals only. However, since most of the actual movies are in color, it lacks sufficient theoretical basis. In this study, we formulate the problem of applying various transformations corresponding to luminance, density, and gamma-corrected values in the discretized domain to the CIE XYZ color space. By using the asymptotic expansion method in the continuous-valued domain to these respective values, we compare the required quantization accuracy with that obtained by the corresponding analysis in discretized domain. The results show that the required number of quantization bits obtained by the analysis in the discretized domain is consistent with the results of computer simulations and the approximations obtained in the continuous-valued domain for the cases of luminance (14 bits), density (12 bits), and gamma-corrected value quantization (11 bits), respectively. The required accuracy of conventional TV signals, which have a smaller signal range than that of digital cinema, is also investigated, and its validity is confirmed by the simulation. The proposed analysis method makes it possible to obtain the required number of quantization bits without time-consuming computer simulations.

Key Words: digital cinema, digital TV, required number of bits, color difference, XYZ signal, CIE 1976 $L^*a^*b^*$

1. INTRODUCTION

In recent years, there has been rapid progress in improving the quality of imaging systems, such as digital cinema [1],[2] that can achieve quality exceeding HDTV for moving images, and archiving systems for art and natural history images [3] for still images. There are several evaluation measures to determine the image quality in such high-quality imaging systems, including resolution, frequency transfer characteristics, and SNR [4]. Among them, one of the most important measures, regardless of whether the system is for video or still image system, is the required quantization accuracy of the image signal. Various studies have been targeted for the problem of quantization accuracy, but there are many obstacles such as the need for a huge amount of computer simulations to obtain useful design values. Therefore, if an analytical method that formulates the required quantization accuracy based on some visual model is available, then it is expected to contribute to the simple design of imaging systems for various applications.

The color space of the digital cinema standard specified by the Digital Cinema Initiative (DCI) in the U.S. is capable of supporting future display

* Department of Computer Science, Faculty of Applied Information Science, Hiroshima Institute of Technology
** School of Information Science and Technology, Aichi Prefectural University (retired)

devices with a wide color gamut by inputting and storing images by the form of CIE XYZ signals [5],[6]. Therefore, it is desirable to study the required quantization accuracy for CIE XYZ signals. The required number of quantization bits for color signals is determined so that quantized step changes do not make any artifacts invisible in the continuous signals before quantization [7]-[10]. This is similar to the method of obtaining the required number of quantization bits for monochrome images based on luminance signal [11]-[17].

Although there is a controversy on the evaluation formula for color difference, we will focus on the CIE 1976 *L*a*b** color difference, which is widely used and recommended by the CIE. Our previous study [8] have analyzed the required quantization accuracy by approximating the quantization error as a continuous function under the assumption that the quantization step size is sufficiently small, but it cannot be applied for gamma values smaller than 1 when the gamma-corrected value is used, and also it is impossible to identify the sample points that give maximum color difference. In this paper, we make use of the same color difference analysis as the previous one in the discretized region to derive the exact solution, and compare it with the results obtained by the approximate solution.

In section 2, we show the color difference formula and its variable transformations, and obtain the quantization step size from the analytical model of color difference. Section 3 shows the behavior of the transform function for a system with a relatively wide signal range (i.e., the minimum luminance value is extremely small), such as digital cinema. In Section 4, the behavior of the transform function is shown for a system such as digital TV, which has a narrower dynamic range than the digital cinema. In section 5, we derive points where the transform functions simultaneously reach their maximum color difference. In section 6, numerical examples of the theoretical values are presented and their validity is verified by computer simulations. Section 7 gives conclusions and future work.

2. COLOR DIFFERENCE FORMULA AND ITS VARIABLE TRANSFORMATION

In the following discussions, x, y, and z are normalized signals of X, Y, and Z with their white points of the maximum values X_0, Y_0, and Z_0, respectively, and let $x(m_1)$, $y(m_2)$, and $z(m_3)$ be their sample points quantized with M bits. Here, m_1, m_2, and m_3 are integers that exist in the range determined by the number of quantization bits:

$$0 \le m_1, m_2, m_3 \le 2^M - 1. \tag{1}$$

2.1 QUANTIZATION FOR VARIOUS TRANSFORM

The minimum values of the luminance x, y, and z are set to ρ. The quantized values $x(m_1)$, $y(m_2)$, and $z(m_3)$ for the luminance, density, and gamma-corrected transform are considered in turn below when these values are all uniformly quantized with M bits. Here, m is used as an integer value to represent any of m_1, m_2, and m_3 in Eq. (1). Assuming that the uniform quantization functions for luminance, density, and gamma-corrected value quantization are $V_a(m)$, $V_b(m)$, and $V_c(m)$, and that these quantization step sizes are Δ_a, Δ_b, and Δ_c, respectively, they are represented as follows:

$$V_a(m) = \rho + \Delta_a m, \quad \Delta_a = (1-\rho) \Big/ \left(2^M - 1\right), \tag{2}$$

$$V_b(m) = \rho \cdot 10^{\Delta_b m}, \quad \Delta_b = -\log(\rho) \Big/ \left(2^M - 1\right), \tag{3}$$

$$V_c(m) = \rho + \Delta_c^{\gamma} m^{\gamma}, \quad \Delta_c = (1-\rho)^{1/\gamma} \Big/ \left(2^M - 1\right). \tag{4}$$

Then the quantized values of $x(m_1)$, $y(m_2)$, and $z(m_3)$ are obtained by applying above transform functions to m_1, m_2, and m_3 [8].

2.2 COLOR DIFFERENCE BY QUANTIZATION

To evaluate the color difference caused by quantization in the CIE 1976 *L*a*b** color space, we make use of the following function $f(w)$ of a luminance value w [8]:

$$f(w) = \begin{cases} w^{1/3}, & \text{for} \quad \theta < w \le 1, \\ \phi(w + 2\theta)/3, & \text{for} \quad \rho \le w \le \theta. \end{cases} \tag{5}$$

When $\rho \le w \le \theta$, the function satisfies

$$f(w) = \phi(w + 2\theta)/3, \tag{6}$$

where $\theta = (24/116)^3$, $\phi = (116/24)^2$.

The color difference caused by quantization is evaluated between adjacent quantization samples. Therefore, if we consider a sample point at m_1, m_2, and m_3 as the reference for quantization values, then the adjacent quantization values are those that have changed by +1 or −1 from at least one coordinate value. Among those 26 adjacent samples, the color difference due to quantization has a maximum value when the change is, from the results of analysis in the continuous domain, either $m_1 \rightarrow m_1-1$, $m_2 \rightarrow m_2+1$, and $m \rightarrow m_3-1$, or $m_1 \rightarrow m_1+1$, $m_2 \rightarrow m_2-1$, and $m_3 \rightarrow m_3+1$. We prove that the results obtained from the analysis in the continuous domain also hold in the discrete domain.

If we denote the sample points where at least one of m_1, m_2, and m_3 changes by Δm_1, Δm_2, and Δm_3 (where Δm_1, Δm_2, and Δm_3 are either −1, 0, or +1, except when they are all zero at the same time), the 26 adjacent sample points can be represented by Figure 1 in 3-D $m_1 m_2 m_3$ space. Here, the neighboring sample points are categorized to three sets in terms of the number of differences in the coordinate values where groups 1 to 3 correspond to all-axis-value change (8 samples), 2-axis-value change (12 samples), 1-axis-value change (6 samples), respectively, and the samples are indexed within each group. For example, the sample point represented by (2-7) is the seventh point in group 2. Table 1 summarizes Δm_1, Δm_2, and Δm_3 values for each sample point.

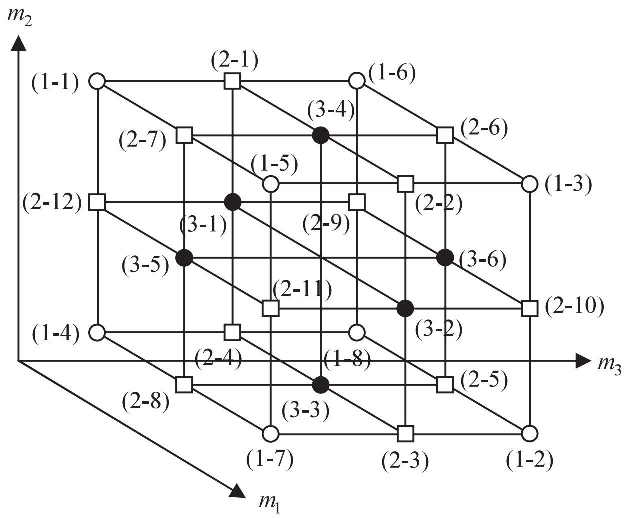


Fig. 1 Adjacent samples in 3-D $m_1 m_2 m_3$ space.

Table 1 Δm_1, Δm_2, and Δm_3 values for each sample point.

Point	Δm_1	Δm_2	Δm_3	Point	Δm_1	Δm_2	Δm_3
Group 1							
(1-1)	−1	+1	−1	(1-5)	+1	+1	−1
(1-2)	+1	−1	+1	(1-6)	−1	+1	+1
(1-3)	+1	+1	+1	(1-7)	+1	−1	−1
(1-4)	−1	−1	−1	(1-8)	−1	−1	+1
Group 2							
(2-1)	−1	+1	0	(2-7)	0	+1	−1
(2-2)	+1	+1	0	(2-8)	0	−1	−1
(2-3)	+1	−1	0	(2-9)	+1	0	−1
(2-4)	−1	−1	0	(2-10)	+1	0	+1
(2-5)	0	−1	+1	(2-11)	−1	0	+1
(2-6)	0	+1	+1	(2-12)	−1	0	−1
Group 3							
(3-1)	−1	0	0	(3-4)	0	+1	0
(3-2)	+1	0	0	(3-5)	0	0	−1
(3-3)	0	−1	0	(3-6)	0	0	+1

The color difference in CIE 1976 $L^*a^*b^*$ color space between the reference and the adjacent sample points is denoted by $\Delta E(m_1, m_2, m_3)$. The function V represents one of V_a, V_b, or V_c.

$$\begin{aligned}&\{\Delta E(m_1, m_2, m_3)\}^2\\ &= A\left[f\{V(m_2+\Delta_{m2})\} - f\{V(m_2)\}\right]^2\\ &+B\left\langle\left[f\{V(m_1+\Delta_{m1})\} - f\{V(m_1)\}\right]-\left[f\{V(m_2+\Delta_{m2})\} - f\{V(m_2)\}\right]\right\rangle^2\\ &+C\left\langle\left[f\{V(m_2+\Delta_{m2})\} - f\{V(m_2)\}\right]-\left[f\{V(m_3+\Delta_{m3})\} - f\{V(m_3)\}\right]\right\rangle^2,\end{aligned} \tag{7}$$

where $A = 116^2$, $B = 500^2$, $C = 200^2$. If we define $\psi(m)$ as

$$\psi(m) = f\{V(m+\Delta m)\} - f\{V(m)\}, \tag{8}$$

then $\{\Delta E(m_1, m_2, m_3)\}^2$ can be shown as

$$\begin{aligned}&\{\Delta E(m_1, m_2, m_3)\}^2\\ &= A\{\psi(m_2)\}^2 + B\{\psi(m_1)-\psi(m_2)\}^2 + C\{\psi(m_2)-\psi(m_3)\}^2.\end{aligned} \tag{9}$$

In addition, m_1, m_2, and m_3 are transformed into m_1', m_2', and m_3' according to Δm_1, Δm_2, and Δm_3 values.

$$\left.\begin{aligned} m_i' &= m_i + 1, \quad \text{for} \quad \Delta_{mi} = 1,\\ m_i' &= m_i, \quad\quad \text{for} \quad \Delta_{mi} = -1.\end{aligned}\right\} \quad (i = 1, 2, 3) \tag{10}$$

Then, m_i' varies from 1 to $2^M - 1$. The function $\psi(m_1')$ can be represented as

$$\left.\begin{aligned}\psi(m_1') &= f\{V(m_1')\} - f\{V(m_1'-1)\} = U(m_1') \quad \text{for} \quad \Delta_{m1} = 1,\\ \psi(m_1') &= f\{V(m_1'-1)\} - f\{V(m_1')\} = -U(m_1') \quad \text{for} \quad \Delta_{m1} = -1,\end{aligned}\right\} \tag{11}$$

where $U(m)$ is defined by

$$U(m) = f\{V(m)\} - f\{V(m-1)\}, \quad 1 \le m \le 2^M - 1. \tag{12}$$

Similarly, $\psi(m_2')$ and $\psi(m_3')$ can be represented as

$$\left.\begin{aligned}\psi(m_2') &= U(m_2'), \quad \psi(m_3') = U(m_3') \quad\quad \text{for} \quad \Delta_{m2} = 1,\ \Delta_{m3} = 1,\\ \psi(m_2') &= -U(m_2'), \quad \psi(m_3') = -U(m_3') \quad \text{for} \quad \Delta_{m2} = -1,\ \Delta_{m3} = -1.\end{aligned}\right\} \tag{13}$$

Therefore, by putting $e\{U(m_1'), U(m_2'), U(m_3')\}$ for $\{\Delta E(m_1, m_2, m_3)\}^2$, which is the square of the color difference of the reference sample against its 26 adjacent samples, we can analyze their functional behavior. The analysis is applied to 26 sample points denoted (1-1) to (3-6) shown in Figure 1. It can be seen that the function h is isometrically largest if the following condition holds.

$$U(m_1'), U(m_2'), U(m_3') > 0, \quad 1 \le m_1', m_2', m_3' \le 2^M - 1. \tag{14}$$

The function $e\{U(m_1'), U(m_2'), U(m_3')\}$ also monotonically increases, and it has a maximum value at the upper bound.

3. BEHAVIOR OF $U(m)$ FOR $\rho < \theta$

3.1 CLASSIFICATION OF $U(m)$'s FORM

In the case of quantization of signals with various transforms, if we let m_θ be the value of m corresponding to θ that determines the classification of the function $f(w)$, then m_θ is generally a real number. The form of $U(m)$ differs at three regions on the line of m shown in Figure 2 as follows.

$$U(m) = \begin{cases} \phi\{V(m) - V(m-1)\}/3, & \text{for } 1 \le m \le \lfloor m_\theta \rfloor, \\ V(m)^{1/3} - \phi\{V(m-1) + 2\theta\}/3, & \text{for } m = \lfloor m_\theta \rfloor + 1, \\ V(m)^{1/3} - V(m-1)^{1/3}, & \text{for } \lfloor m_\theta \rfloor + 2 \le m \le 2^M - 1. \end{cases} \tag{15}$$

The minimum and maximum values of $U(m)$ in each region are denoted by $\min[U(m)]$ and $\max[U(m)]$, respectively, in the following discussion.

3.2 BEHAVIOR FOR LUMINANCE QUANTIZATATION

If we let $m_\theta^{(L)}$ be m_θ for luminance quantization as

$$m_\theta^{(L)} = (\theta - \rho)/\Delta_a, \tag{16}$$

then, $U(m)$ is positive in the region of $1 \le m \le \lfloor m_\theta^{(L)} \rfloor$:

$$U(m) = \phi\Delta_a/3 > 0. \tag{17}$$

For $m = \lfloor m_\theta^{(L)} \rfloor + 1$, $U(m)$ is

$$U(m) = \left\{\rho + \Delta_a\left(\lfloor m_\theta^{(L)} \rfloor + 1\right)\right\}^{1/3} - \phi\left\{\left(\rho + \Delta_a\lfloor m_\theta^{(L)} \rfloor\right) + 2\theta\right\}/3, \tag{18}$$

and it becomes larger than 0 by using the relation

$$\lfloor m_\theta^{(L)} \rfloor < m_\theta^{(L)} < \lfloor m_\theta^{(L)} \rfloor + 1. \tag{19}$$

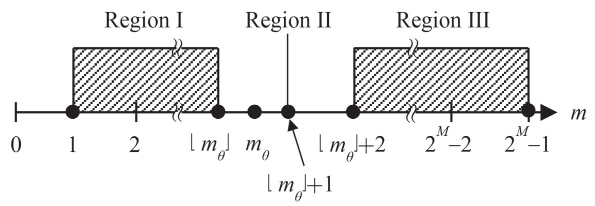


Fig. 2 Domain separation in m for function $U(m)$.

So that $U(m)$ is positive in the region II. Finally, for $\lfloor m_\theta^{(L)} \rfloor + 2 \le m \le 2^M - 1$, $U(m)$ and its min/max values can be shown as

$$U(m) = V_a(m)^{1/3} - V_a(m-1)^{1/3}, \tag{20}$$

$$\min[U(m)] = 1 - (1 - \Delta_a)^{1/3} > 0, \tag{21}$$

$$\max[U(m)] = \left\{\rho + \Delta_a\left(\lfloor m_\theta^{(L)} \rfloor + 2\right)\right\}^{1/3} - \left\{\rho + \Delta_a\left(\lfloor m_\theta^{(L)} \rfloor + 1\right)\right\}^{1/3}. \tag{22}$$

As a result, $U(m)$ is positive in the region III.

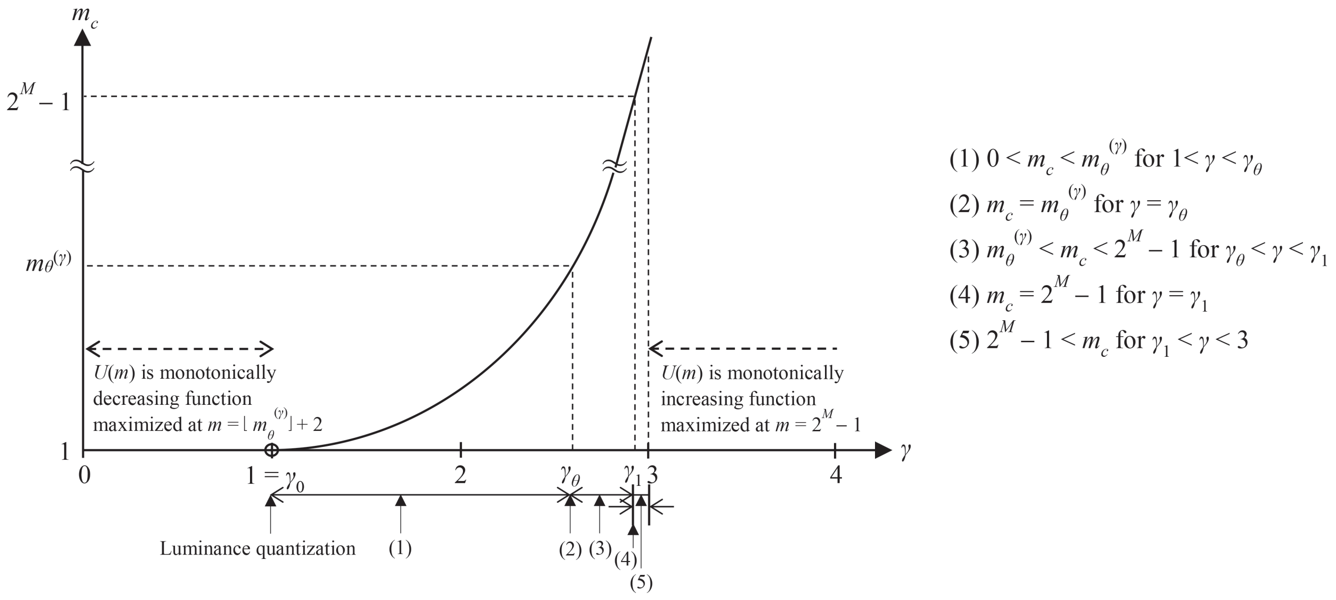


Fig. 3 γ-dependency of m_c that gives maximum value of $h(m)$ in $\lfloor m_\theta^{(\gamma)} \rfloor + 2 < m < 2^M - 1$.

3.3 BEHAVIOR FOR DENSITY QUANTIZATATION

Let $m_\theta(D)$ be m_θ for density quantization as

$$m_\theta^{(D)} = (1/\Delta_b)\log(\theta/\rho). \quad (23)$$

Then, $U(m)$ and its min/max values in the region of $1 \le m \le \lfloor m_\theta^{(D)} \rfloor$ can be shown as follows.

$$U(m) = \phi\{V_b(m) - V_b(m-1)\}/3, \quad (24)$$

$$\min[U(m)] = \phi\rho\left(10^{\Delta_b} - 1\right)/3 > 0, \quad (25)$$

$$\max[U(m)] = \phi\rho 10^{\Delta_b\left(\lfloor m_\theta^{(D)} \rfloor - 1\right)}\left(10^{\Delta_b} - 1\right)/3 > 0. \quad (26)$$

As a result, $U(m)$ is positive in the region I. Next, $U(m)$ in the region of $m = \lfloor m_\theta^{(D)} \rfloor + 1$ is

$$U(m) = \left\{\rho \cdot 10^{\Delta_b\left(\lfloor m_\theta^{(D)} \rfloor + 1\right)}\right\}^{1/3} - \phi\left\{\rho \cdot 10^{\Delta_b \lfloor m_\theta^{(D)} \rfloor} + 2\theta\right\}/3. \quad (27)$$

We can see that $U(m)$ in the region II is positive by applying

$$\lfloor m_\theta^{(D)} \rfloor < m_\theta^{(D)} < \lfloor m_\theta^{(D)} \rfloor + 1. \quad (28)$$

Finally, $U(m)$ and its min/max values in the region of $\lfloor m_\theta^{(D)} \rfloor + 2 \le m \le 2^M - 1$ are given as follows.

$$U(m) = \rho^{1/3}\left(10^{\Delta_b/3} - 1\right) \cdot 10^{\Delta_b(m-1)/3}, \quad (29)$$

$$\min[U(m)] = \rho^{1/3} \cdot \left(10^{\Delta_b/3} - 1\right) \cdot 10^{\Delta_b/3} \cdot 10^{\Delta_b \lfloor m_\theta^{(D)} \rfloor/3} > 0, \quad (30)$$

$$\max[U(m)] = 1 - 10^{-\Delta_b/3} > 0, \quad (31)$$

As a result, $U(m)$ is positive in the region III.

3.4 BEHAVIOR FOR GAMMA-CORRECTED VALUE QUANTIZATATION

Let $m_\theta^{(\gamma)}$ be m_θ for gamma-corrected value quantization as

$$m_\theta^{(\gamma)} = (\theta - \rho)^{1/\gamma}/\Delta_c. \quad (32)$$

Then, $U(m)$ in the region of $1 \le m \le \lfloor m_\theta^{(\gamma)} \rfloor$ is

$$U(m) = \phi\{V_c(m) - V_c(m-1)\}/3. \quad (33)$$

$U(m)$ is constant for $\gamma = 1$ (corresponding luminance quantization), and is monotonically increasing function for $\gamma > 1$, while monotonically decreasing and positive function for $0 < \gamma < 1$. The constant values of $U(m)$ for $\gamma = 1$ are given by

$$U(m) = \phi\Delta_c^\gamma/3 = \phi\Delta_c/3, \quad (34)$$

$$\Delta_c = (1-\rho)^{1/\gamma}/\left(2^M - 1\right) = (1-\rho)/\left(2^M - 1\right), \quad (35)$$

and min/max values of $U(m)$ for $0 < \gamma < 1$ are given by

$$\min[U(m)] = \phi\Delta_c^\gamma\left\{\left(\lfloor m_\theta^{(\gamma)} \rfloor\right)^\gamma - \left(\lfloor m_\theta^{(\gamma)} \rfloor - 1\right)^\gamma\right\}/3 > 0, \quad (36)$$

$$\max[U(m)] = \phi\Delta_c^\gamma/3 > 0. \quad (37)$$

Next, $U(m)$ in the region of $m = \lfloor m_\theta^{(\gamma)} \rfloor + 1$ is

$$U(m) = \left\{\rho + \Delta_c^\gamma\left(m_\theta^{(\gamma)} + 1\right)^\gamma\right\}^{1/3} - (1/3)\phi\left[\left\{\rho + \Delta_c^\gamma\left(\lfloor m_\theta^{(\gamma)} \rfloor\right)^\gamma\right\} + 2\theta\right]. \quad (38)$$

It can be seen that $U(m)$ is positive by applying the relation

$$\lfloor m_\theta^{(\gamma)} \rfloor < m_\theta^{(\gamma)} < \lfloor m_\theta^{(\gamma)} \rfloor + 1. \quad (39)$$

Finally, $U(m)$ in the region of $\lfloor m_\theta^{(\gamma)} \rfloor + 2 \le m \le 2^M - 1$ is

$$U(m) = V_c(m)^{1/3} - V_c(m-1)^{1/3}. \quad (40)$$

As it is difficult to clarify the behavior of $U(m)$ analytically, we consider the next functions $h(m)$ and $g(m)$ instead of $U(m)$.

$$h(m) = \left(\rho + \Delta_c^\gamma m^\gamma\right)^{-2/3} m^{\gamma - 1}, \quad (41)$$

$$g(m) = \left(\Delta_c^\gamma/3\right)(\gamma - 3)m^\gamma + \rho(\gamma - 1). \quad (42)$$

It is not difficult to see that the function $g(m)$ determines the sign of $d\{h(m)\}/dm$. If m_c is zero for $g(m)$, then the m_c gives a maximum value of $h(m)$, and m_c values depend on γ as shown in Figure 3. The behavior of $U(m)$ for $0 < \gamma < 1$ and $3 \le \gamma$ is shown in the same figure. Table 2 summarizes $U(m)$ and its min/max values in the region of $\lfloor m_\theta^{(\gamma)} \rfloor + 2 \le m \le 2^M - 1$.

The values of $U(m)$ in Table 2 at $m = \lfloor m_\theta^{(\gamma)} \rfloor + 2$, $2^M - 1$, $\lfloor m_c \rfloor$, $\lfloor m_c \rfloor + 1$, and $\lfloor m_c \rfloor + 2$ are respectively given by

$$U\left(\lfloor m_\theta^{(\gamma)} \rfloor + 2\right) = \left\{\rho + \Delta_c^\gamma\left(\lfloor m_\theta^{(\gamma)} \rfloor + 2\right)^\gamma\right\}^{1/3} - \left\{\rho + \Delta_c^\gamma\left(\lfloor m_\theta^{(\gamma)} \rfloor + 1\right)^\gamma\right\}^{1/3} > 0, \quad (43)$$

$$U\left(2^M - 1\right) = 1 - \left[\rho + (1-\rho)\left\{1 - 1/\left(2^M - 1\right)\right\}^{1/\gamma}\right]^{1/3} > 0, \quad (44)$$

$$U\left(\lfloor m_c \rfloor\right) = \left\{\rho + \Delta_c^\gamma\left(\lfloor m_c^{(\gamma)} \rfloor\right)^\gamma\right\}^{1/3} - \left\{\rho + \Delta_c^\gamma\left(\lfloor m_c^{(\gamma)} \rfloor - 1\right)^\gamma\right\}^{1/3} > 0, \quad (45)$$

$$U\left(\lfloor m_c \rfloor + 1\right) = \left\{\rho + \Delta_c^\gamma\left(\lfloor m_c^{(\gamma)} \rfloor + 1\right)^\gamma\right\}^{1/3} - \left\{\rho + \Delta_c^\gamma\left(\lfloor m_c^{(\gamma)} \rfloor\right)^\gamma\right\}^{1/3} > 0, \quad (46)$$

$$U\left(\lfloor m_c \rfloor + 2\right) = \left\{\rho + \Delta_c^\gamma\left(\lfloor m_c^{(\gamma)} \rfloor + 2\right)^\gamma\right\}^{1/3} - \left\{\rho + \Delta_c^\gamma\left(\lfloor m_c^{(\gamma)} \rfloor + 1\right)^\gamma\right\}^{1/3} > 0. \quad (47)$$

As a result, it can be seen that $U(m)$ in the region III is positive because both $U(\lfloor m_\theta^{(\gamma)} \rfloor + 2)$ and $U(2^M - 1)$, which give a minimum value of $U(m)$, are positive.

Table 2 Behavior of $U(m)$ for $\lfloor m_\theta^{(\gamma)} \rfloor + 2 \le m \le 2^M - 1$.

γ	$0 < \gamma < 1$, $1 < \gamma < \gamma_\theta$, $\gamma = \gamma_\theta$	$\gamma_\theta < \gamma < \gamma_1$	$\gamma = \gamma_1$, $\gamma_1 < \gamma < 3$, $\gamma = 3$, $3 < \gamma$
$U(m)$	monotonically decreasing	maximized around $m = m_c$	monotonically increasing
$m_{\max}$	$m = \lfloor m_\theta \rfloor + 2$	$m = \lfloor m_c \rfloor + 1$ or $m = \lfloor m_c \rfloor + 2$ for $h(\lfloor m_c \rfloor + 1) > h(\lfloor m_c \rfloor)$ $m = \lfloor m_c \rfloor$ or $m = \lfloor m_c \rfloor + 1$ for $h(\lfloor m_c \rfloor + 1) < h(\lfloor m_c \rfloor)$	$m = 2^M - 1$
$m_{\min}$	$m = 2^M - 1$	$m = \lfloor m_\theta^{(\gamma)} \rfloor + 2$ or $m = 2^M - 1$	$m = \lfloor m_\theta \rfloor + 2$

(*) $m_{\max}$ and $m_{\min}$ give maximum and minimum values of $U(m)$, respectively. $\gamma_\theta = 3(\theta/\rho)\{2+(\theta/\rho)\}$, $\gamma_1 = 3(1/\rho)\{2+(1/\rho)\}$ [8].

4. BEHAVIOR OF $U(m)$ FOR $\theta \le \rho$

In the previous section, we investigated the behavior of the function $U(m)$ for relatively wide rage signal (i.e., $\rho < \theta$). This section studies the behavior of $U(m)$ for relatively narrow range signal (i.e., $\theta \le \rho$).

4.1 FUNCTIONAL FORM OF $U(m)$

The functional form of $U(m)$ for $\theta \le \rho$ is given as follows similar to the region III of $\theta \le \rho$ in Figure 2.

$$U(m) = V(m)^{1/3} - V(m-1)^{1/3}. \quad (48)$$

4.2 BEHAVIOR FOR LUMINANCE QUANTIZATATION

When the function V in Eq. (48) is replaced by V_a in Eq. (2), the form of $U(m)$ is

$$U(m) = V_a(m)^{1/3} - V_a(m-1)^{1/3}. \quad (49)$$

Then, $U(m)$ is monotonically decreasing function with respect to m, and it has a minimum value at $m = 2^M - 1$, and has a maximum value at $m = 1$, so that they are given as follows.

$$\min[U(m)] = 1 - (1 - \Delta_a)^{1/3} > 0, \quad (50)$$

$$\max[U(m)] = \{\rho + \Delta_a\}^{1/3} - \rho^{1/3} > 0. \quad (51)$$

As a result, $U(m)$ is positive in this case.

4.3 BEHAVIOR FOR DENSITY QUANTIZATATION

When the function V in Eq. (48) is replaced by V_b in Eq. (3), the form of $U(m)$ is

$$U(m) = \rho^{1/3}\left(10^{\Delta_b/3} - 1\right) \cdot 10^{\Delta_b(m-1)/3}, \quad (52)$$

Then, $U(m)$ is monotonically increasing function with respect to m, and it has a minimum value at $m = 1$, and has a maximum value at $m = 2^M - 1$, so that they are given as follows.

$$\min[U(m)] = \rho^{1/3} \cdot \left(10^{\Delta_b/3} - 1\right) > 0. \quad (53)$$

$$\max[U(m)] = 1 - 10^{-\Delta_b/3} > 0. \quad (54)$$

As a result, $U(m)$ is positive in this case.

4.4 BEHAVIOR FOR GAMMA-CORRECTED VALUE QUANTIZATATION

For the gamma-corrected value quantization, the

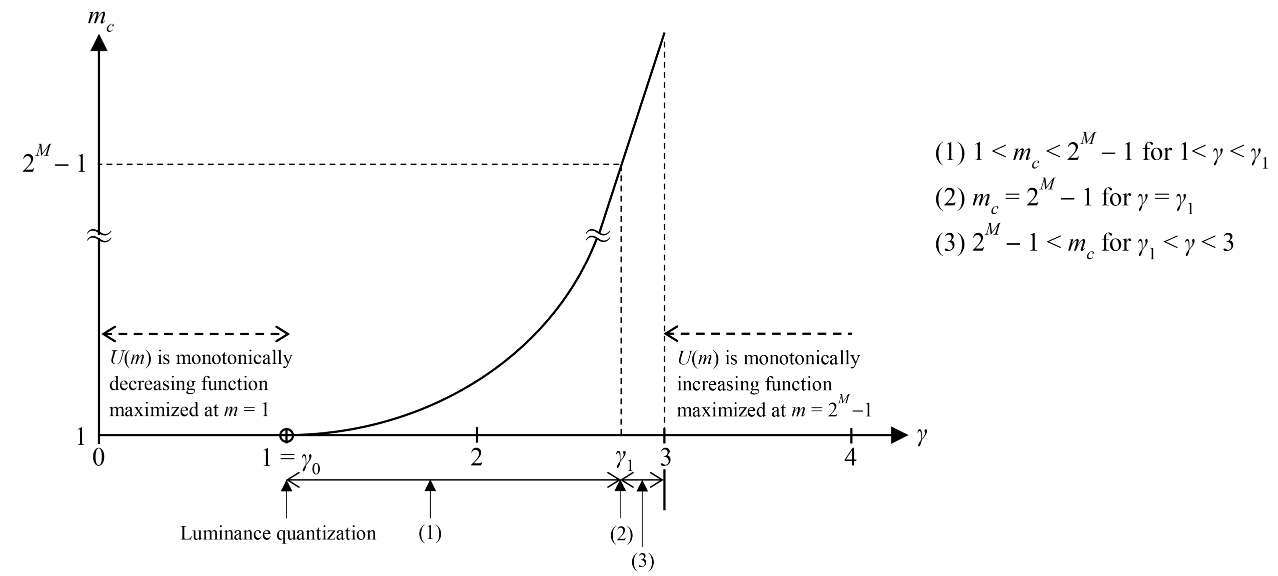


Fig. 4 γ-dependency of m_c that gives maximum value of $h(m)$ for $\theta \le \rho$

form of $U(m)$ is

$$U(m) = V_c(m)^{1/3} - V_c(m-1)^{1/3}. \quad (55)$$

as the function V in Eq. (48) is replaced by V_c in Eq. (4). The behavior of $U(m)$ is as described in 3. except the domain is $1 \le m \le 2^M - 1$ instead of $\lfloor m_\theta^{(\gamma)} \rfloor + 2 \le m \le 2^M - 1$, so that the min/max values are different. The γ-dependency of m_c, which gives a maximum value of $h(m)$ in eq. (41) is shown in Figure 4. $U(m)$ and its min/max values for $\theta \le \rho$ are summarized in Table 3.

It is obvious that $U(m)$ has its maximum value of $U(\lfloor m_c \rfloor + 1)$ or $U(\lfloor m_c \rfloor + 2)$ for $h(\lfloor m_c \rfloor + 1) > h(\lfloor m_c \rfloor)$, and $U(\lfloor m_c \rfloor)$ or $U(\lfloor m_c \rfloor + 1)$ for $h(\lfloor m_c \rfloor + 1) < h(\lfloor m_c \rfloor)$. The minimum value of $U(m)$ at $m = 1$ and $2^M - 1$ is obtained as follows, and these values are both positive.

$$U(1) = \left(\rho + \Delta_c^\gamma\right)^{1/3} - \rho^{1/3} > 0, \quad (56)$$

$$U\left(2^M - 1\right) = 1 - \left[\rho + (1-\rho)\left\{1 - 1/\left(2^M - 1\right)\right\}^\gamma\right]^{1/3} > 0. \quad (57)$$

As a result, U(m) is positive in this case.

Table 3 Behavior of $U(m)$ for $\theta \le \rho$

γ	$0 < \gamma < 1$, $1 < \gamma < \gamma_1$, $\gamma = \gamma_1$	$\gamma_1 < \gamma < 3$	$\gamma = 3, \gamma < 3$
$U(m)$	monotonically decreasing	maximized around $m = m_c$	monotonically increasing
$m_{\max}$	$m = 1$	$m = \lfloor m_c \rfloor + 1$ or $m = \lfloor m_c \rfloor + 2$ for $h(\lfloor m_c \rfloor + 1) > h(\lfloor m_c \rfloor)$ $m = \lfloor m_c \rfloor$ or $m = \lfloor m_c \rfloor + 1$ for $h(\lfloor m_c \rfloor + 1) < h(\lfloor m_c \rfloor)$	$m = 2^M - 1$
$m_{\min}$	$m = 2^M - 1$	$m = 1$ or $m = 2^M - 1$	$m = 1$

(*) $m_{\max}$ and $m_{\min}$ give maximum and minimum values of $U(m)$, respectively. $\gamma_1 = 3(1/\rho)\{2+(1/\rho)\}$ [7].

5. MAXIMUM COLOR DIFFERENCE

5.1 REQUIRED NUMBER OF QUANTIZATION BITS

The previous discussion proves that $U(m)$ is positive for all luminance, density, and gamma-corrected value quantization against any values within the range specified by Eq. (1) for both conditions $\rho < \theta$ and $\theta \le \rho$. Therefore, it can be seen that $e\{U(m_1'), U(m_2'), U(m_3')\}$, the square of the color difference, gives the maximum value for coordinate changes of $m_1 \to m_1 - 1$, $m_2 \to m_2 + 1$, and $m_3 \to m_3 - 1$, together with $m_1 \to m_1 + 1$, $m_2 \to m_2 - 1$, and $m_3 \to m_3 + 1$. In addition, the square of this color difference takes its maximum value at the point where $U(m_1')$, $U(m_2')$, and $U(m_3')$ have simultaneously their maximum values because the partial differentiations with respect to $U(m_1')$, $U(m_2')$, and $U(m_3')$ are all positive.

In this section, we derive the point where $U(m_1)$, $U(m_2)$, and $U(m_3)$ simultaneously reach their maximum in each region of $m_1m_2m_3$ space in the cases of $m_1 \to m_1 - 1$, $m_2 \to m_2 + 1$, and $m_3 \to m_3 - 1$, and calculate $e\{U(m_1'), U(m_2'), U(m_3')\}$ at that point. The maximum value of $e\{U(m_1'), U(m_2'), U(m_3')\}$ in each region gives the maximum color difference of the whole, and the required number of quantization bits is determined from the condition that the maximum color difference does not exceed 1.

5.2 DOMAIN OF m_1, m_2, m_3 AND ITS CLASSIFICATION

Let each quantization step size be given by

$$\Delta_{m1} = -1, \quad \Delta_{m2} = 1, \quad \Delta_{m3} = -1. \quad (58)$$

Then, the relation between m_i' and m_i ($i = 1, 2, 3$) is

$$m_1' = m_1, \quad m_2' = m_2 + 1, \quad m_3' = m_3. \quad (59)$$

The ranges of m_1, m_2, and m_3 have been defined in Eq. (1). Since we only need to find the color difference by limiting the change of quantization points in the $m_1m_2m_3$ space to either 0 or ±1, the set of quantized samples to be considered contains 3 × 3 × 3 = 27 points. Figure 5 shows the domain of m_1, m_2, and m_3 in the 3-D space for $\theta \le \rho$, where the quantization points are indicated by the numbers enclosed in square symbols. Table 4 shows the domain of m_1, m_2, and m_3 in each region.

5.3 QUANTIZED POINTS FOR COLOR DIFFERENCE COMPUTATION

The candidate points that possibly give the maximum color difference can be determined in 3-D $m_1m_2m_3$ space for $\rho < \theta$. Figure 6 and 7 shows these points for luminance and density quantization, respectively. Note that m_1 and m_3 are identical to m itself, while m_2 is subtracted by 1 from m according to Eq. (59). In the case of gamma-corrected value

Table 4 Domain of m_1, m_2, and m_3 in each region for $\rho < \theta$.

Region	Domain m_2	Domain m_1	Domain m_3
1	$0 \le m_2 \le \lfloor m_\theta \rfloor - 1$	$m_1 = \lfloor m_\theta \rfloor + 1$	$m_3 = \lfloor m_\theta \rfloor + 1$
2		$\lfloor m_\theta \rfloor + 2 \le m_1 \le 2^M - 1$	$m_3 = \lfloor m_\theta \rfloor + 1$
3		$\lfloor m_\theta \rfloor + 2 \le m_1 \le 2^M - 1$	$1 \le m_3 \le \lfloor m_\theta \rfloor$
4		$m_1 = \lfloor m_\theta \rfloor + 1$	$1 \le m_3 \le \lfloor m_\theta \rfloor$
5		$1 \le m_1 \le \lfloor m_\theta \rfloor$	$1 \le m_3 \le \lfloor m_\theta \rfloor$
6		$1 \le m_1 \le \lfloor m_\theta \rfloor$	$m_3 = \lfloor m_\theta \rfloor + 1$
7		$1 \le m_1 \le \lfloor m_\theta \rfloor$	$\lfloor m_\theta \rfloor + 2 \le m_3 \le 2^M - 1$
8		$m_1 = \lfloor m_\theta \rfloor + 1$	$\lfloor m_\theta \rfloor + 2 \le m_3 \le 2^M - 1$
9		$\lfloor m_\theta \rfloor + 2 \le m_1 \le 2^M - 1$	$\lfloor m_\theta \rfloor + 2 \le m_3 \le 2^M - 1$
10	$m_2 = \lfloor m_\theta \rfloor$	$m_1 = \lfloor m_\theta \rfloor + 1$	$m_3 = \lfloor m_\theta \rfloor + 1$
11		$\lfloor m_\theta \rfloor + 2 \le m_1 \le 2^M - 1$	$m_3 = \lfloor m_\theta \rfloor + 1$
12		$\lfloor m_\theta \rfloor + 2 \le m_1 \le 2^M - 1$	$1 \le m_3 \le \lfloor m_\theta \rfloor$
13		$m_1 = \lfloor m_\theta \rfloor + 1$	$1 \le m_3 \le \lfloor m_\theta \rfloor$
14		$1 \le m_1 \le \lfloor m_\theta \rfloor$	$1 \le m_3 \le \lfloor m_\theta \rfloor$
15		$1 \le m_1 \le \lfloor m_\theta \rfloor$	$m_3 = \lfloor m_\theta \rfloor + 1$
16		$1 \le m_1 \le \lfloor m_\theta \rfloor$	$\lfloor m_\theta \rfloor + 2 \le m_3 \le 2^M - 1$
17		$m_1 = \lfloor m_\theta \rfloor + 1$	$\lfloor m_\theta \rfloor + 2 \le m_3 \le 2^M - 1$
18		$\lfloor m_\theta \rfloor + 2 \le m_1 \le 2^M - 1$	$\lfloor m_\theta \rfloor + 2 \le m_3 \le 2^M - 1$
19	$\lfloor m_\theta \rfloor + 1 \le m_2 \le 2^M - 2$	$m_1 = \lfloor m_\theta \rfloor + 1$	$m_3 = \lfloor m_\theta \rfloor + 1$
20		$\lfloor m_\theta \rfloor + 2 \le m_1 \le 2^M - 1$	$m_3 = \lfloor m_\theta \rfloor + 1$
21		$\lfloor m_\theta \rfloor + 2 \le m_1 \le 2^M - 1$	$1 \le m_3 \le \lfloor m_\theta \rfloor$
22		$m_1 = \lfloor m_\theta \rfloor + 1$	$1 \le m_3 \le \lfloor m_\theta \rfloor$
23		$1 \le m_1 \le \lfloor m_\theta \rfloor$	$1 \le m_3 \le \lfloor m_\theta \rfloor$
24		$1 \le m_1 \le \lfloor m_\theta \rfloor$	$m_3 = \lfloor m_\theta \rfloor + 1$
25		$1 \le m_1 \le \lfloor m_\theta \rfloor$	$\lfloor m_\theta \rfloor + 2 \le m_3 \le 2^M - 1$
26		$m_1 = \lfloor m_\theta \rfloor + 1$	$\lfloor m_\theta \rfloor + 2 \le m_3 \le 2^M - 1$
27		$\lfloor m_\theta \rfloor + 2 \le m_1 \le 2^M - 1$	$\lfloor m_\theta \rfloor + 2 \le m_3 \le 2^M - 1$

quantization, these candidate points are shown in Figure 8 and 9 as examples for cases of $0 < \gamma < 1$ and $\gamma_1 < \gamma$, respectively. The points for other conditions of gamma values are readily found by examining the behavior of the color difference function shown in Table 2.

The points that give the maximum color difference with $\theta \le \rho$ for various quantization are shown in 3-D $m_1m_2m_3$ space in Figure 10. In this case, color differences for luminance and density quantization are maximized at $m = 1$ and $m = 2^M - 1$, respectively. The color difference for gamma-corrected value quantization is determined according to its behavior shown in Table 3.

6. NUMERICAL EXAMPLES

6.1 CONSTANTS

We use $\rho = 10^{-3.2}$ and $10^{-2.0}$, which are typical for cinema film screens [3] and television monitors [4], respectively. As already shown in Eq. (6), $\theta = (24/116)^3 \approx 0.00885645$ and $\phi = (116/24)^2 \approx 23.36111111$. Thus, the cases of $\rho = 10^{-3.2}$ and $\rho = 10^{-2.0}$ corresponds to $\rho < \theta$ as discussed in 3. and $\theta \le \rho$ as discussed in 4., respectively.

The following values of $m_\theta^{(L)}$, $m_\theta^{(D)}$, and $m_\theta^{(\gamma)}$ are used for luminance, density, and gamma-corrected value quantization. Note that these constants are not defined when the minimum value of luminance is $\rho = 10^{-2.0}$, since $\theta \le \rho$ in this case.

$$m_\theta^{(L)} = \left[\left\{(24/116)^3 - 10^{-3.2}\right\} \Big/ \left(1 - 10^{-3.2}\right)\right]\left(2^M - 1\right) \approx 0.00823069\left(2^M - 1\right), \tag{60}$$

$$m_\theta^{(D)} = (1/3.2) \times \log\left\{(24/116)^3 \big/ 10^{-3.2}\right\}\left(2^M - 1\right) \approx 0.35851867\left(2^M - 1\right), \tag{61}$$

$$m_\theta^{(\gamma)} = \left[\left\{(24/116)^3 - 10^{-3.2}\right\} \Big/ \left(1 - 10^{-3.2}\right)\right]^{1/\gamma}\left(2^M - 1\right) \approx 0.00823069^{1/\gamma}\left(2^M - 1\right). \tag{62}$$

γ_θ and γ_1 are given as follows

$$\gamma_\theta = 3\theta/(\theta + 2\rho) = 3(24/116)^3 \Big/ \left\{(24/116)^3 + 2 \times 10^{-3.2}\right\} \approx 2.62585422, \tag{63}$$

$$\gamma_1 = 3\big/(1 + 2\rho) = 3\big/\left(1 + 2 \times 10^{-3.2}\right) \approx 2.99621903. \tag{64}$$

6.2 REQUIRED NUMBER OF BITS

In numerical evaluations, the required number of quantization bits is obtained by calculating the color difference ΔE and applying the condition that it does not exceed 1. Substituting M = 14 into Eq. (61) for luminance quantization, $m_\theta^{(L)} \approx 134.843$ and $\lfloor m_\theta^{(L)} \rfloor = 135$ are obtained. Figure 11 (a) shows the result of ΔE for $\rho = 10^{-3.2}$. The point 5 in the figure gives maximum color difference ΔE_{max} = 0.514562 at (m_1, m_2, m_3) = (134, 133, 134).

In the case of density quantization, by substituting M = 12 into Eq. (62), $m_\theta^{(D)} \approx 1468.13$ and $\lfloor m_\theta^{(D)} \rfloor = 1468$ are obtained. Figure 11 (b) shows the result of ΔE for $\rho = 10^{-3.2}$. The point 27 in the figure gives ΔE_{max} = 0.649522 at (m_1, m_2, m_3) = (4095, 4094, 4095).

In the case of gamma-corrected value quantization, $m_\theta^{(\gamma)}$ and $\lfloor m_\theta^{(\gamma)} \rfloor$ depend on both γ and M. Figure 11 (c) shows the result of ΔE for $\rho = 10^{-3.2}$ and $\gamma = 2.6$. The point 27 in the figure gives ΔE_{max} = 0.558289 at (m_1, m_2, m_3) = (325, 324, 325). On the other hand, Figure 11 (d) shows the result of ΔE for $\rho = 10^{-3.2}$ and $\gamma = 2.8$. The point 46 in the figure gives ΔE_{max} = 0.53105 at (m_1, m_2, m_3) = (479, 478, 479).

Table 5 shows maximum color differences between the quantized points and their adjacent points calculated from the computer simulation only for critical situations that the maximum color difference changes from greater than 1 to less than 1. It is confirmed from the simulation that the maximum color difference for luminance quantization is obtained at $m = 1$ in Eq. (17), while that for density quantization is obtained at $m = 2^M - 1$ in Eq. (31) In addition, the maximum color difference for gamma-corrected value quantization is determined by the value of m in Table 2 which depends on γ as described in **3.4**. These results show that the theoretical values agree with the computer-simulated values, and therefore the validity of the theory has been verified.

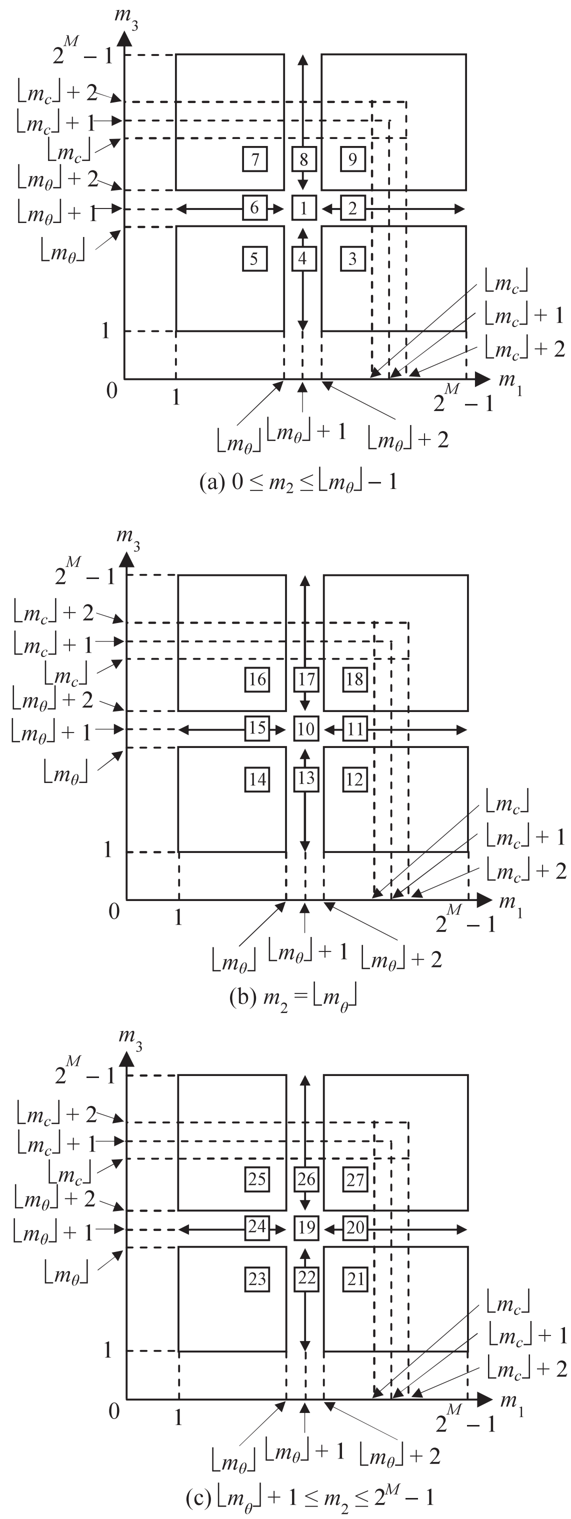
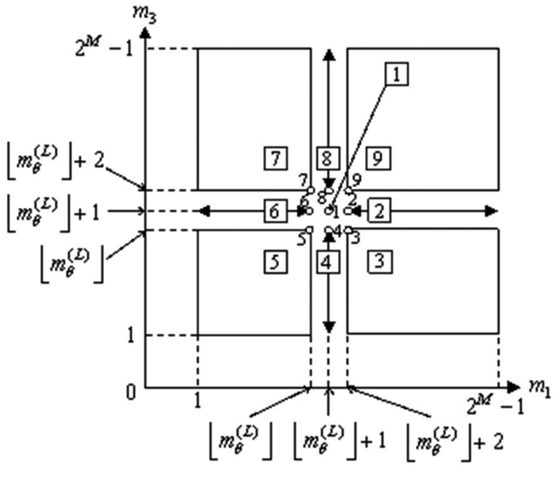
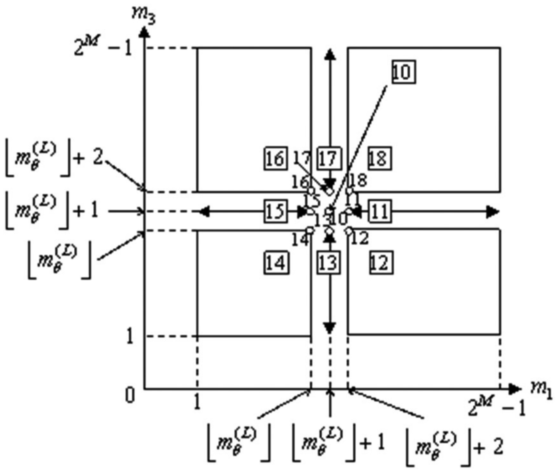


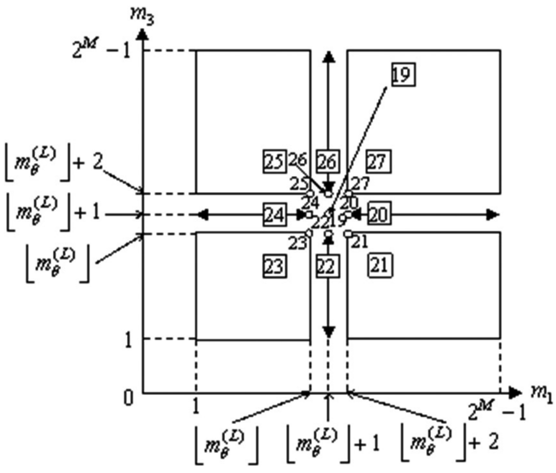
Fig. 5 Domain of m_1, m_2, and m_3 in 3-D space for $\theta \le \rho$.



(a) $0 \le m_2 \le \lfloor m_\theta^{(L)} \rfloor - 1$

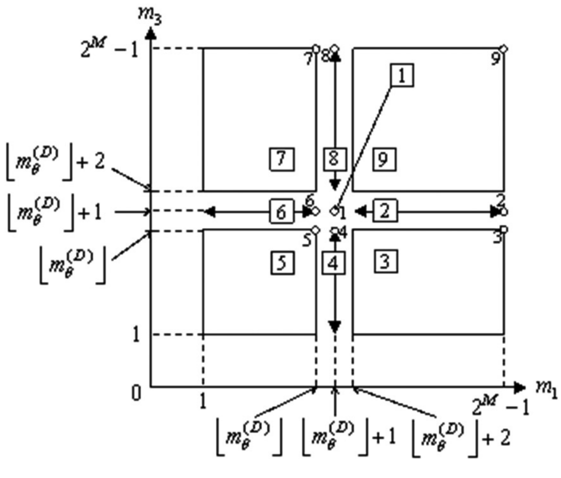


(b) $m_2 = \lfloor m_\theta^{(L)} \rfloor$

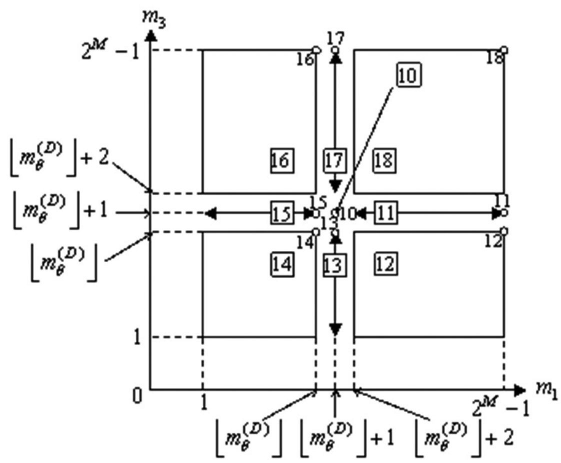


(c) $\lfloor m_\theta^{(L)} \rfloor + 1 \le m_2 \le 2^M - 1$

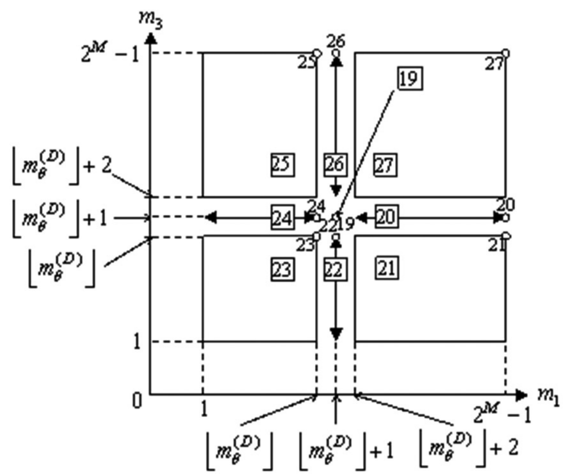
Fig. 6 Candidate points for giving maximum color difference in luminance quantization ($\rho < \theta$). ΔE has a maximum value at (a) $m_2 = \lfloor m_\theta^{(L)} \rfloor - 1$, and (b) $m_2 = \lfloor m_\theta^{(L)} \rfloor + 1$ for every point.



(a) $0 \le m_2 \le \lfloor m_\theta^{(D)} \rfloor - 1$

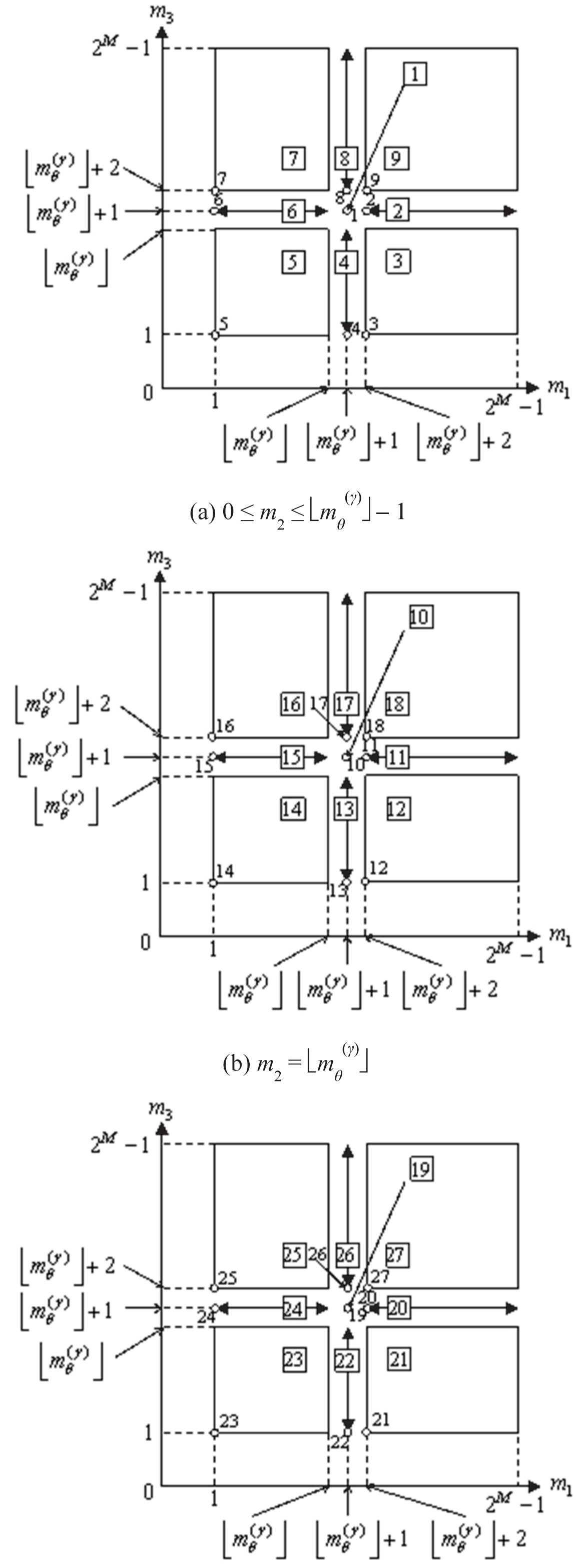


(b) $m_2 = \lfloor m_\theta^{(D)} \rfloor$



(c) $\lfloor m_\theta^{(D)} \rfloor + 1 \le m_2 \le 2^M - 1$

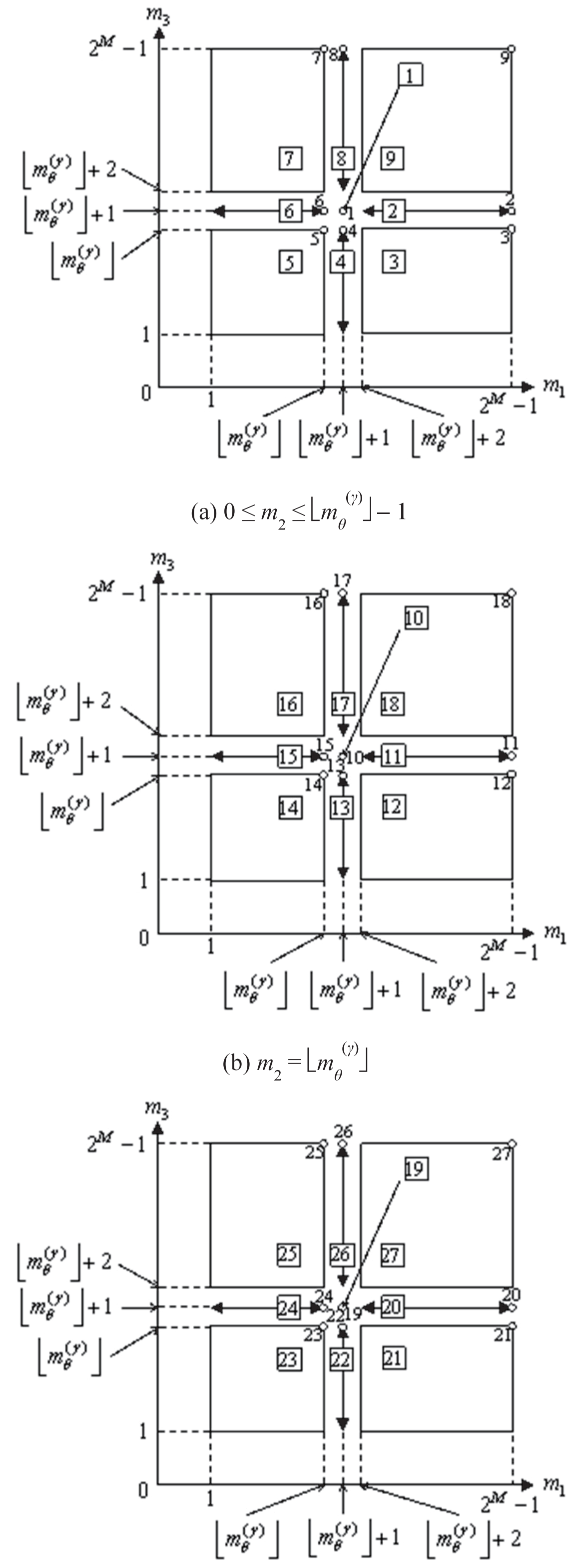
Fig. 7 Candidate points for giving maximum color difference in density quantization ($\rho < \theta$). ΔE has a maximum value at (a) $m_2 = \lfloor m_\theta^{(D)} \rfloor - 1$, and (b) $m_2 = 2^M - 2$ for every point.



(a) $0 \le m_2 \le \lfloor m_\theta^{(\gamma)} \rfloor - 1$

(b) $m_2 = \lfloor m_\theta^{(\gamma)} \rfloor$

(c) $\lfloor m_\theta^{(\gamma)} \rfloor + 1 \le m_2 \le 2^M - 1$



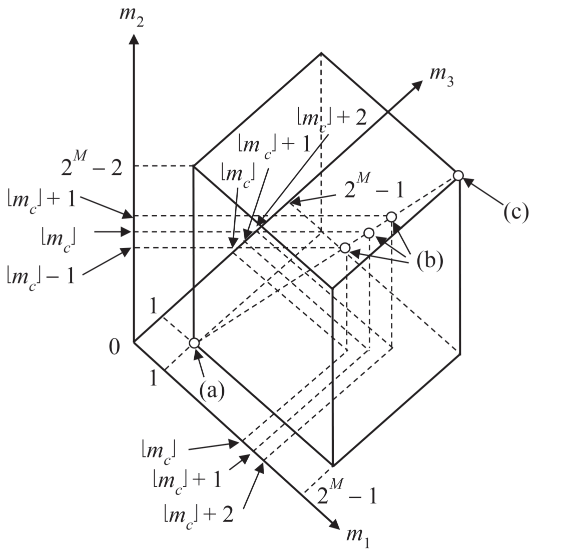
(a) $0 \le m_2 \le \lfloor m_\theta^{(\gamma)} \rfloor - 1$

(b) $m_2 = \lfloor m_\theta^{(\gamma)} \rfloor$

(c) $\lfloor m_\theta^{(\gamma)} \rfloor + 1 \le m_2 \le 2^M - 1$

Fig. 8 Candidate points for giving maximum color difference in gamma-corrected value quantization ($\rho < \theta$, $0 < \gamma < 1$). ΔE has a maximum value at (a) $m_2 = 0$, and (b) $m_2 = \lfloor m_\theta^{(\gamma)} \rfloor + 1$ for every point.

Fig. 9 Candidate points for giving maximum color difference in gamma-corrected value quantization ($\rho < \theta$, $\gamma_1 < \gamma$). ΔE has a maximum value at (a) $m_2 = \lfloor m_\theta^{(\gamma)} \rfloor - 1$, and (b) $m_2 = 2^M - 2$ for every point.



(a) Luminance and gamma-corrected quantization ($0 < \gamma < 1$)
(b) Gamma-corrected quantization ($0 < \gamma < \gamma_1$)
(c) Density and gamma-corrected quantization ($\gamma_1 < \gamma$)

Fig. 10 Points for calculating color difference in 3-D $m_1m_2m_3$ space for $\theta \leq \rho$.

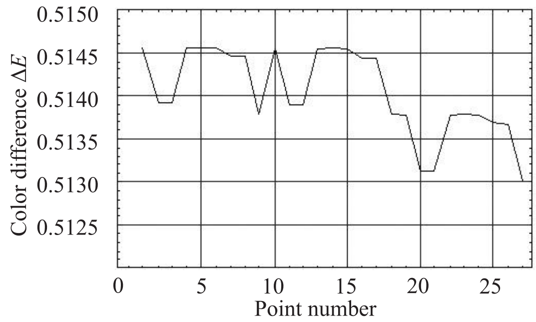
7. CONCLUTION

The proposed analysis method for calculating color differences in the discretized region has the following advantages over that in the continuous approximation.

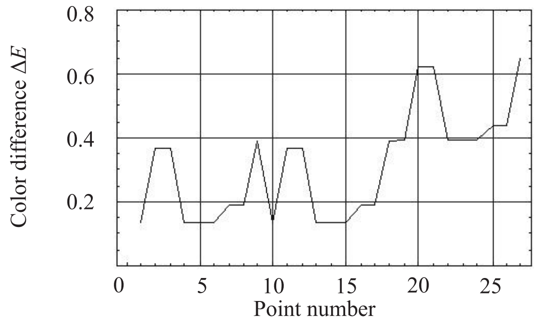
- The analysis can be applied even when the quantization step size is not sufficiently small.
- It is possible to analyze the case of $0 < \gamma < 1$ in gamma-corrected value quantization, which cannot be handled by the continuous approximation analysis.
- It is possible to identify the exact quantized sample coordinates that give the maximum color difference.

On the other hand, the analysis of the approximations for continuous signals has the advantage that the required number of quantization bits can be obtained by calculating only one single equation.

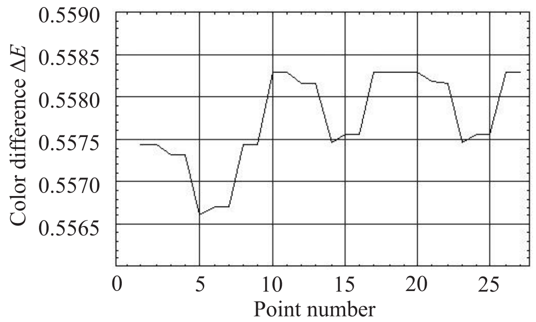
Future work includes confirming the effectiveness of this analysis method in more appropriate color difference formulas and color spaces, and studying contrast and gamut mapping algorithms that take into account the three-dimensional distribution of color differences due to quantization.



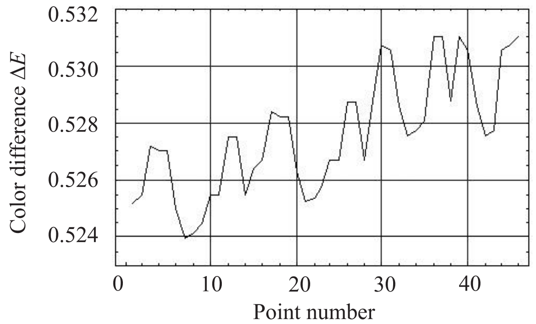
(a) Luminance quantization for $\rho = 10^{-3.2}$, $M = 14$



(b) Density quantization for $\rho = 10^{-3.2}$, $M = 12$



(c) Gamma-corrected value quantization for $\rho = 10^{-3.2}$, $\gamma = 2.6$, $M = 11$



(d) Gamma-corrected value quantization for $\rho = 10^{-3.2}$, $\gamma = 2.8$, and $M = 11$

Fig. 11 Color difference at quantized points.

Table 5 Discrete sample coordinates and maximum color difference values where ΔE_{max} changes from greater than 1 to less than 1. (In numerical evaluations)

Quantization	D_r	γ	M	Quantized points	ΔE_{max}
Luminance quantization	3.2		13	(1, 0, 1), (0, 1, 0)	1.029189
			14	(1, 0, 1), (0, 1, 0)	0.514570
	2.0		12	(1, 0, 1), (0, 1, 0)	1.865776
			13	(1, 0, 1), (0, 1, 0)	0.936488
Density quantization	3.2		11	(2047, 2046, 2047), (2046, 2047, 2046)	1.298971
			12	(4095, 4094, 4095), (4094, 4095, 4094)	0.649522
	2.0		10	(1023, 1022, 1023), (1022, 1023, 1022)	1.624263
			11	(2047, 2046, 2047), (2046, 2047, 2046)	0.812040
Gamma-corrected value quantization	3.2	1.5	11	(84,83,84), (83,84,83)	1.245516
			12	(167,166,167), (166,167,166)	0.622649
		2.6	10	(163, 162, 163), (162, 163, 162)	1.117097
			11	(325, 324, 325), (324, 325, 324)	0.558298
		2.7	10	(191,190,191), (190,191,190)	1.084915
			11	(382,381,382), (381,382,381)	0.542193
		2.8	10	(239, 238, 239), (238, 239, 239)	1.062618
			11	(479, 478, 479), (478, 479, 478)	0.531050
		3.0	10	(1023,1022,1023), (1022,1023,1022)	1.048308
			11	(2047,2046,2047), (2046,2047,2046)	0.523900
		3.5	10	(1023,1022,1023), (1022,1023,1022)	1.234510
			11	(2047,2046,2047), (2046,2047,2046)	0.616979
	2.0	1.5	10	(48,47,48), (47,48,47)	1.537771
			11	(96,95,96), (95,96,95)	0.768519
		2.6	9	(227, 226, 227), (226, 227, 226)	1.934337
			10	(455, 454, 455), (454, 455, 454)	0.966224
		2.7	9	(267,266,267), (266,267,266)	1.953849
			10	(533,532,533), (532,533,532)	0.975970
		2.8	9	(322, 321, 322), (321, 322, 321)	1.985180
			10	(644, 643, 644), (643, 644, 643)	0.991620
		3.0	10	(1023,1022,1023), (1022,1023,1022)	1.048308
			11	(2047,2046,2047), (2046,2047,2046)	0.523900
		3.5	10	(1023,1022,1023), (1022,1023,1022)	1.222924
			11	(2047,2046,2047), (2046,2047,2046)	0.611192

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