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**Article**

# Linear invariant families on the homogeneous ball of a complex Banach space

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**Abstract**

We study the notion of linear invariance on the unit ball of a JB*-triple $X$, and we obtain some connection between the norm-order of a linear invariant family and the starlikeness of order 1/2. Also, we give some result concerning the radius of univalence of some linear invariant families.

**Key Words:** JB*-triple, linear invariant family, norm order

## 1 Introduction

Recently, several interesting results, concerning the norm-order of a linear invariant family and some connections with starlikeness, convexity and other geometric properties of holomorphic mappings in $\mathbb{C}^n$, were obtained by Pfaltzgraff and Suffridge [23]. Also they showed a number of growth, covering and distortion results for mappings that belong to a linear invariant family on the Euclidean unit ball in $\mathbb{C}^n$. Hamada and Kohr generalized the results in [23] to the unit ball in a complex Hilbert space in [9] and to the unit polydisc in [10]. For linear invariant families in several complex variables, see also the books [3, 4] and the references therein.

This paper is concerned with the study of linear invariance on the homogeneous ball of a complex Banach space. A complex Banach space is a JB*-triple if, and only if, its open unit ball is homogeneous. All four types of classical Cartan domains and their infinite dimensional analogues are the open unit balls of JB*-triples, and the same holds for any finite or infinite product of these domains ([13], see also [8, 15]). Thus the unit balls of JB*-triples are natural generalizations of the unit disc in $\mathbb{C}$ and we have a setting in which a large number of bounded symmetric homogeneous domains may be studied simultaneously. We obtain some connection between the norm-order of a linear invariant family and the starlikeness of order 1/2. Also, we give some result concerning the radius of univalence of some linear invariant families.

## 2 Preliminaries

Let $B$ be the unit ball in a complex Banach space $X$. Let $Y$ be a complex Banach space. A holomorphic mapping $f : B \to Y$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in B$. A holomorphic mapping $f : B \to Y$ is said to be biholomorphic if $f(B)$ is a domain in $Y$, $f^{-1}$ exists and holomorphic on $f(B)$. A biholomorphic mapping $f : B \to Y$ is said to be convex if $f(B)$ is a convex domain. Let $X^*$ be the dual space of $X$. For each $x \in X \setminus \{0\}$, we define

$$T(x) = \{x^* \in X^* : \|x^*\| = 1, x^*(x) = \|x\|\}.$$

By the Hahn-Banach theorem, $T(x)$ is nonempty. Let $f : B \to X$ be a locally biholomorphic mapping. Let $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$. We say that $f$ is a starlike mapping of order $\alpha$ if

$$\left| \frac{1}{\|x\|} x^* \left( [Df(x)]^{-1} f(x) \right) - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}$$

for $x \in B \setminus \{0\}$, $x^* \in T(x)$.

Let $L(X, Y)$ denote the set of continuous linear operators from $X$ into $Y$. Let $I_X$ be the identity in $L(X, X)$.

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Let $\mathcal{LS}(B)$ denote the family of locally biholomorphic mappings from $B$ to $X$, normalized by $f(0)=0$ and $Df(0)=I_X$.

We recall that a JB*-triple is a complex Banach space $X$ together with a continuous mapping (called Jordan triple product)

$$X\times X\times X\to X \qquad (x,y,z)\mapsto\{x,y,z\}$$

such that for all elements in $X$ the following conditions $(\mathrm{J}_1)$-$(\mathrm{J}_4)$ hold, where for every $x,y\in X$, the operator $x\square y$ on $X$ is defined by $z\mapsto\{x,y,z\}$:

$(\mathrm{J}_1)$ $\{x,y,z\}$ is symmetric bilinear in the outer variable $x$, $z$ and conjugate linear in the inner variable $y$,

$(\mathrm{J}_2)$ $\{a,b,\{x,y,z\}\}$
$=\{\{a,b,x\},y,z\}-\{x,\{b,a,y\},z\}+\{x,y,\{a,b,z\}\}$,
(Jordan triple identity)

$(\mathrm{J}_3)$ $x\square x\in L(X,X)$ is a hermitian operator with spectrum $\geqq 0$,

$(\mathrm{J}_4)$ $\|\{x,x,x\}\|=\|x\|^3$.

It is known [16, p.523] that in this definition condition $(\mathrm{J}_4)$ can be replaced by $\|x\square x\|=\|x\|^2$ and that

$$\|x\square y\|\le\|x\|\cdot\|y\|$$

holds for all $x,y\in X$. Then, we have

$$\|\{x,y,z\}\|\le\|x\|\cdot\|y\|\cdot\|z\|, \quad \text{for all } x,y,z. \tag{2.1}$$

**Example 2.1.** *Let $S$ be a locally compact topological space and let $C_0(S)$ be the Banach space of all continuous complex valued functions $f$ on $S$ vanishing at infinity with $\|f\|=\sup|f(S)|$. Then $C_0(S)$ is a JB*-triple with $\{f,g,h\}=f\bar{g}h$.*

A linear subspace $I\subset X$ is called a *subtriple* if $\{I,I,I\}\subset I$.

For every $a\in X$, let $Q_a:X\to X$ be the conjugate linear operator defined by $Q_a(x)=\{a,x,a\}$. This operator is called the quadratic representation and it satisfies the fundamental formula

$$Q_{Q_a(b)}=Q_aQ_bQ_a$$

for all $a,b\in X$. For every $x,y\in X$, the Bergman operator $B(x,y)\in L(X,X)$ is defined by

$$B(x,y)=I_X-2x\square y+Q_xQ_y.$$

From (2.1), we have

$$\|B(x,y)\|\le(1+\|x\|\cdot\|y\|)^2, \quad x,y\in X. \tag{2.2}$$

In case $\|x\square y\|<1$, the spectrum of $B(x,y)$ lies in $\{z\in\mathbb{C}:|z-1|<1\}$. In particular, the fractional power $B(x,y)^r\in GL(X)$ exists for every $r\in\mathbb{R}$ in a natural way (cf. [16, p.517]).

Let $B$ be the unit ball of a JB*-triple $X$. Then, for each $a\in B$, the Möbius transformation $g_a$ defined by

$$g_a(x)=a+B(a,a)^{1/2}(I_X+x\square a)^{-1}x, \tag{2.3}$$

is a biholomorphic mapping of $B$ onto itself with $g_a(0)=a$, $g_a(-a)=0$ and $g_{-a}=g_a^{-1}$.

**Proposition 2.2.** *Let $g_a$ be as above. Then for any $a\in B$, $g_a$ extends biholomorphically to a neighborhood of $\bar{B}$ and we have*

$$[Dg_a(0)]^{-1}D^2g_a(0)(x,y)=-2\{x,a,y\}, \tag{2.4}$$

$$\|Dg_a(0)\|\le 1, \tag{2.5}$$

$$\|[Dg_a(0)]^{-1}\|=\frac{1}{1-\|a\|^2}, \tag{2.6}$$

$$Dg_{\zeta a}(0)=Dg_a(0), \quad |\zeta|=1, \tag{2.7}$$

$$g_a(a)=\frac{2}{1+\|a\|^2}a, \tag{2.8}$$

$$g_a(x)=x+a-\{x,a,x\}+O(\|a\|^2), \tag{2.9}$$

$$[Dg_a(0)]^{-1}=I_X+O(\|a\|^2). \tag{2.10}$$

*Moreover, we have*

$$\frac{1}{1-\|g_{-z}(w)\|^2}\le\frac{(1+\|w\|\cdot\|z\|)^2}{(1-\|w\|^2)(1-\|z\|^2)}, \quad z,w\in B. \tag{2.11}$$

*Proof.* Since $\|x\square a\|\le\|x\|\cdot\|a\|$, $g_a$ and $g_a^{-1}=g_{-a}$ extend holomorphically to $\|x\|<1/\|a\|$. Then, $g_a$ extends biholomorphically to a neighborhood of $\bar{B}$. Since

$$\begin{aligned} g_a(x)&=a+B(a,a)^{1/2}[x-(x\square a)x]+O(\|x\|^3)\\ &=a+B(a,a)^{1/2}[x-\{x,a,x\}]+O(\|x\|^3), \end{aligned}$$

we have

$$Dg_a(x)(y)=B(a,a)^{1/2}[y-\{y,a,x\}-\{x,a,y\}]+O(\|x\|^2)$$

and

$$\begin{aligned} D^2g_a(0)(y,z)&=B(a,a)^{1/2}[-\{y,a,z\}-\{z,a,y\}]\\ &=-2B(a,a)^{1/2}\{y,a,z\}. \end{aligned}$$

Since $Dg_a(0) = B(a,a)^{1/2}$, we obtain (2.4). By [17, Corollary 3.6], we obtain (2.5) and (2.6). Since

$$B(\zeta a,\zeta a) = B(a,a), \quad |\zeta| = 1,$$

we obtain (2.7). Since the JB*-subtriple of $X$ generated by $a$, denoted by $X_a$, is isometrically isomorphic to $\mathcal{C}_0(S)$ for some locally compact subset $S \subset \mathbb{R}$ ([16]), it is easy to see that in $X_a$ and hence in $X$, we have

$$g_a(a) = \frac{2}{1+\|a\|^2}a.$$

Thus, we obtain (2.8). Since $B(a,a)^{1/2} = I_X + O(\|a\|^2)$, we have (2.10) and

$$g_a(x) = a + B(a,a)^{1/2}[x - \{x,a,x\}] + O(\|a\|^2)$$
$$= a + x - \{x,a,x\} + O(\|a\|^2).$$

Since

$$\frac{1}{1-\|g_{-z}(w)\|^2} = \|B(w,w)^{-1/2}B(w,z)B(z,z)^{-1/2}\|, \tag{2.12}$$

$z,w \in B$ by [19, Proposition 3.1], we obtain (2.11) from (2.2) and (2.6). □

$x \in X$ is called *regular* if $x\square x \in GL(X)$ and $x \in X$ is called a *tripotent* if $\{x,x,x\} = x$. A point $u \in \bar{B}$ is said to be a *real* (*resp. complex*) *extreme point of* $\bar{B}$ if the only $x \in X$ satisfying $\|u + \lambda x\| \le 1$ for all real (resp. complex) numbers $\lambda$ with $|\lambda| \le 1$ is $x = 0$. We call $u \in \bar{B}$ *holomorphically extreme in* $\bar{B}$ if for every open neighborhood $U$ of $0 \in \mathbb{C}$ and every holomorphic mapping $f: U \to X$ the conditions $f(0) = u$ and $f(U) \subset \bar{B}$ imply that $f'(0) = 0$. $u \in \partial B$ is called a *simple boundary point of B* if $u + ty \in \partial B$, $y \in X$, $t \in \mathbb{C}$, $|t| < 1$ always implies $y = 0$. The following result is obtained in Kaup and Upmeier [18, Proposition 3.5].

**Proposition 2.3.** *Let B be the unit ball of a JB*-triple X and $u \in X$. Then the following conditions are equivalent.*

(*i*) *u is a regular tripotent in X*;

(*ii*) *u is holomorphically extreme in* $\bar{B}$;

(*iii*) *u is a complex extreme point of* $\bar{B}$;

(*iv*) *u is a simple boundary point of B.*

Let $\mathcal{E}$ be the set of all complex extreme points of $\bar{B}$. As a corollary of the above proposition, we obtain the following maximum principle for holomorphic mappings on the unit ball of a JB*-triple. When $B$ is the unit ball of a J*-algebra, see Harris [13, Theorem 9]. By the Krein-Milman theorem (see e.g. [5, Chapter 4]), it is known that if $\bar{B}$ is a compact subset of $X$, then $\mathcal{E}$ is nonempty.

**Proposition 2.4.** *Let B be the unit ball of a JB*-triple X and let $\mathcal{E}$ denote the set of all complex extreme points of $\bar{B}$. If $\mathcal{E} \neq \emptyset$, then*

*(i) Let $g_a \in$ Aut$(B)$ given in* (2.3). *Then $g_a(\mathcal{E}) = \mathcal{E}$ for any $a \in B$;*

*(ii) Let Y be a complex Banach space. Let $f: B \to Y$ be a holomorphic mapping with a continuous and bounded extension to $B \cup \mathcal{E}$. Then*

$$\|f(x)\| \le \sup\{\|f(u)\| : u \in \mathcal{E}\}, \quad x \in B.$$

*Moreover, f is completely determined by its value on $\mathcal{E}$.*

*Proof.* (i) Since $g_a^{-1} = g_{-a}$, it suffices to show that $g_a(\mathcal{E}) \subset \mathcal{E}$ for any $a \in B$. Let $v = g_a(u)$, where $u \in \mathcal{E}$. Assume that $v + \lambda x \in \bar{B}$ for $|\lambda| \le 1$. Let

$$h(\lambda) = g_a^{-1}(v + \lambda x), \quad \lambda \in U.$$

Then $h$ is holomorphic on $U$ by Proposition 2.2, $h(0) = g_a^{-1}(v) = u$ and $h(U) \subset \bar{B}$. Since $u$ is a holomorphic extreme point by Proposition 2.3, we must have $h'(0) = 0$. This implies that $Dg_a^{-1}(v)(x) = 0$. Since $g_a^{-1}$ extends biholomorphically to a neighborhood of $\bar{B}$, we obtain $x = 0$. Thus, $v \in \mathcal{E}$.

(ii) By the mean value property for vector valued holomorphic functions, we obtain

$$f(x) = \frac{1}{2\pi}\int_0^{2\pi} f(g_x(e^{i\theta}u))d\theta,$$

where $u \in \mathcal{E}$. Since $g_x(e^{i\theta}u) \in \mathcal{E}$ for $\theta \in [0,2\pi]$ by (i), we obtain (ii). □

## 3 Linear invariance in $X$

We define the notion of linear invariant families and the norm-order in the unit ball $B$ of a complex Banach space $X$.

**Definition 3.1.** *Let B be the unit ball of a complex Banach space X. Then a family $\mathcal{F}$ is called a linear-invariant family if:*

*(i) $\mathcal{F} \subset \mathcal{LS}(B)$,*

*and*

*(ii) $\Lambda_\phi(f) \in \mathcal{F}$, for all $f \in \mathcal{F}$ and $\phi \in$ Aut$B$,*

*where* Aut$B$ *denotes the set of biholomorphic automorphisms of B, and $\Lambda_\phi(f)$ is the Koebe-transform*

$$\Lambda_\phi(f)(x) = [D\phi(0)]^{-1}[Df(\phi(0))]^{-1}(f(\phi(x)) - f(\phi(0))), \tag{3.1}$$

*for all* $x \in B$.

Note that the Koebe transform has the group property $\Lambda_\psi \circ \Lambda_\phi = \Lambda_{\phi\circ\psi}$.

If $\mathcal{F}$ is a linear invariant family, we define two types of *norm-order* of $\mathcal{F}$ (cf.[23]), given by

$$\| ord \|_{X,1}\, \mathcal{F} = \sup_{f\in\mathcal{F}} \sup_{\|y\|=1} \left\{ \frac{1}{2} \| D^2 f(0)(y,\cdot) \| \right\}$$

and

$$\| ord \|_{X,2}\, \mathcal{F} = \sup_{f\in\mathcal{F}} \sup_{\|y\|=1} \left\{ \frac{1}{2} \| D^2 f(0)(y,y) \| \right\}.$$

It is clear that $\| ord \|_{X,1}\, \mathcal{F} \geq \| ord \|_{X,2}\, \mathcal{F}$. Since

$$\begin{aligned} &D^2 f(0)(y,z) \\ &= \frac{1}{2}\{ D^2 f(0)(y+z,y+z) - D^2 f(0)(y,y) \\ &\quad - D^2 f(0)(z,z) \}, \end{aligned}$$

we obtain $\| ord \|_{X,1}\, \mathcal{F} \leq 3 \| ord \|_{X,2}\, \mathcal{F}$. Moreover, if $X$ is a Hilbert space, then $\| ord \|_{X,1}\, \mathcal{F} = \| ord \|_{X,2}\, \mathcal{F}$ by Hörmander [14, Theorem 4].

We now give some examples of linear invariant families in the unit ball $B$ of a complex Banach space $X$.

**Example 3.2.** *$S(B)$, the set of all biholomorphic mappings in $\mathcal{LS}(B)$. If $X$ is a complex Hilbert space of dimension $n$, where $n > 1$, the linear invariant family $S(B)$ does not have finite norm order (see [23], cf. [1]).*

**Example 3.3.** *$\mathcal{U}_\alpha(B)$, the union of all linear invariant families contained in $\mathcal{LS}(B)$ with norm-order not greater than $\alpha$. This is a generalization of the universal linear invariant families $\mathcal{U}_\alpha = \mathcal{U}_\alpha(\Delta)$ considered in [24].*

**Example 3.4.** *If $\mathcal{G}$ is a nonempty subset of $\mathcal{LS}(B)$, then the linear invariant family generated by $\mathcal{G}$ is the family*

$$\Lambda[\mathcal{G}] = \{\Lambda_\phi(g) : g \in \mathcal{G},\ \phi \in \mathrm{Aut}B\}.$$

The linear invariance is a consequence of the group property of the Koebe transform. Obviously, $\Lambda[\mathcal{G}] = \mathcal{G}$ if and only if $\mathcal{G}$ is a linear-invariant family. In the case of the unit Euclidean ball and the unit polydisc in $\mathbb{C}^n$, this example provided a useful technique for generating many interesting mappings (see [20, 21, 22]). For example, we can use a single mapping $f$ from $\mathcal{LS}(B)$ to generate the linear invariant family $\Lambda[\{f\}]$. The family $\Lambda[\{i\}]$, generated by the identity mapping $i(x) = x$, consists of all the Koebe transforms of $i(x)$.

**Example 3.5.** *$\mathcal{K}(B)$, the set of convex mapping in $\mathcal{LS}(B)$.*

As in the proof of [23, Theorem 5.1], we obtain the following result. We will see later that $\| ord \|_{X,2}\, \mathcal{K}(B) = 1$, if $B$ is the unit ball of a JB*-triple. We remark that if $X = \ell^1$ is the complex Banach space of summable complex sequences, then $\| ord \|_{X,2}\, \mathcal{K}(B) = 0$, since the only mapping $f \in \mathcal{K}(B)$ is the identity mapping [26, Corollary 1].

**Proposition 3.6.** *Let $B$ be the unit ball of a complex Banach space $X$ and let $\mathcal{K}(B)$ be the set of normalized convex mappings on $B$. Then $\| ord \|_{X,2}\, \mathcal{K}(B) \leq 1$.*

When $B$ is the unit ball of a JB*-triple $X$, we have the following first order approximation formula for the Koebe transform of $f$.

**Lemma 3.7.** *Let $g_a \in \mathrm{Aut}(B)$ given in (2.3). If $f \in \mathcal{LS}(B)$, then*

$$\begin{aligned} &[Dg_a(0)]^{-1}[Df(g_a(0))]^{-1}(f(g_a(x)) - f(g_a(0))) \\ &= f(x) + Df(x)(a - \{x,a,x\}) - a - D^2 f(0)(a, f(x)) \\ &\quad + O(\|a\|^2), \quad a \to 0. \end{aligned}$$

*Proof.* Since

$$f(x) = x + \frac{1}{2} D^2 f(0)(x,x) + \frac{1}{6} D^3 f(0)(x,x,x) + \cdots,$$

we have

$$f(g_a(0)) = a + O(\|a\|^2), \tag{3.2}$$

and

$$[Df(g_a(0))]^{-1} = I_X - D^2 f(0)(a,\cdot) + O(\|a\|^2). \tag{3.3}$$

Since

$$f(x+y) = f(x) + Df(x)y + O(\|y\|^2),$$

we obtain from (2.9) that

$$\begin{aligned} f(g_a(x)) &= f(x + a - \{x,a,x\} + O(\|a\|^2)) \\ &= f(x) + Df(x)(a - \{x,a,x\}) + O(\|a\|^2). \end{aligned} \tag{3.4}$$

From (2.10) and (3.3), we have

$$\begin{aligned} &[Dg_a(0)]^{-1}[Df(g_a(0))]^{-1} \\ &= I_X - D^2 f(0)(a,\cdot) + O(\|a\|^2). \end{aligned} \tag{3.5}$$

Then from (3.2), (3.4) and (3.5), we have

$$\begin{aligned} &[Dg_a(0)]^{-1}[Df(g_a(0))]^{-1}(f(g_a(x)) - f(g_a(0))) \\ &= (I_X - D^2 f(0)(a,\cdot))[f(x) + Df(x)(a - \{x,a,x\}) - a] \\ &\quad + O(\|a\|^2) \\ &= f(x) + Df(x)(a - \{x,a,x\}) - a - D^2 f(0)(a, f(x)) \\ &\quad + O(\|a\|^2). \end{aligned}$$

This completes the proof. □

The following useful result is a natural extension to JB*-triples of [24, Lemma 1.2] (cf. [9, 10, 21]).

**Lemma 3.8.** *Let $\mathcal{F}$ be a linear-invariant family on the unit ball $B$ of a JB*-triple $X$ with $\|ord\|_{X,1}\,\mathcal{F}=\alpha$ and $\|ord\|_{X,2}\,\mathcal{F}=\beta$. Then*

$$\alpha=\sup_{f\in\mathcal{F}}\sup\left\{\left\|\frac{1}{2}\Phi(f,x,y,z)-\{y,x,z\}\right\|: \|x\|<1, \|y\|=\|z\|=1\right\}, \tag{3.6}$$

*and*

$$\beta=\sup_{f\in\mathcal{F}}\sup\left\{\left\|\frac{1}{2}\Phi(f,x,y,y)-\{y,x,y\}\right\|: \|x\|<1, \|y\|=1\right\}, \tag{3.7}$$

*where*

$$\Phi(f,x,y,z)=[Dg_x(0)]^{-1}[Df(x)]^{-1}D^2f(x)(Dg_x(0)y,Dg_x(0)z).$$

*Proof.* It is clear that

$$\sup_{f\in\mathcal{F}}\sup\left\{\left\|\frac{1}{2}\Phi(f,x,y,z)-\{y,x,z\}\right\|: \|x\|<1, \|y\|=\|z\|=1\right\}\geq\alpha.$$

On the other hand, let $f\in\mathcal{F}$ and $\phi=g_x$ where $x\in B$. It is clear that $F\in\mathcal{F}$, where $F(w)=\Lambda_\phi(f)(w)$, $w\in B$. Therefore, we have

$$\left\|\frac{1}{2}D^2F(0)(y,z)\right\|\leq\alpha,\quad y,z\in X, \|y\|=\|z\|=1. \tag{3.8}$$

If we differentiate twice the mapping $F=\Lambda_\phi(f)$, given by (3.1), we obtain that

$$DF(w)=[D\phi(0)]^{-1}[Df(\phi(0))]^{-1}Df(\phi(w))D\phi(w),$$

$w\in B$, and

$$\begin{aligned}&D^2F(w)(y,z)\\&=[D\phi(0)]^{-1}[Df(\phi(0))]^{-1}\{D^2f(\phi(w))(D\phi(w)y,D\phi(w)z)\\&\quad+Df(\phi(w))D^2\phi(w)(y,z)\},\quad y,z\in X.\end{aligned}$$

Evaluating at $w=0$, we obtain that

$$\begin{aligned}&D^2F(0)(y,z)\\&=\Phi(f,x,y,z)+[Dg_x(0)]^{-1}D^2g_x(0)(y,z).\end{aligned}$$

Hence, from (2.4) and this equality, we have

$$D^2F(0)(y,z)=\Phi(f,x,y,z)-2\{y,x,z\}.$$

Finally, from (3.8) and the last relation, one concludes that

$$\left\|\frac{1}{2}\Phi(f,x,y,z)-\{y,x,z\}\right\|\leq\alpha,$$

for all $x\in B$ and $y,z\in X$, $\|y\|=\|z\|=1$. Thus, we obtain (3.6). Putting $z=y$ in the above argument, we obtain (3.7). This completes the proof. □

Note that in the case of one complex variable, the relations (3.6) and (3.7) are equivalent to

$$\alpha=\beta=\sup_{f\in\mathcal{F}}\sup_{|b|<1}\left|\frac{1}{2}(1-|b|^2)\frac{f''(b)}{f'(b)}-\bar{b}\right|.$$

(compare with [24, Lemma 1.2]).

Pfaltzgraff and Suffridge [23, Theorem 3.1] proved recently that if $\mathcal{M}$ is a linear invariant family on the Euclidean unit ball of $\mathbb{C}^n$, then $\|ord\|\,\mathcal{M}\geq 1$. Hamada and Kohr obtained the extension of this result to the unit ball of a complex Hilbert space in [9, Theorem 3.2] and to the unit polydisc in [10, Theorem 3.2]. In the following we obtain the extension of this result to the unit ball of a JB*-triple.

**Theorem 3.9.** *Let $\mathcal{F}$ be a linear invariant family on the unit ball $B$ of a JB*-triple $X$. Then $\|ord\|_{X,2}\,\mathcal{F}\geq 1$.*

*Proof.* We will use an argument similar to that in the proof of [23, Theorem 3.1]. Let $\beta=\|ord\|_{X,2}\,\mathcal{F}$ and let $x\in B\setminus\{0\}$ be fixed. Putting $y=\frac{x}{\|x\|}$, $x\in B\setminus\{0\}$, in (3.7), we obtain that

$$\beta\geq\left\|\frac{1}{2\|x\|^2}\Phi(f,x,x,x)-\frac{1}{\|x\|^2}\{x,x,x\}\right\|,$$

where

$$\Phi(f,x,x,x)=[Dg_x(0)]^{-1}[Df(x)]^{-1}D^2f(x)(Dg_x(0)x,Dg_x(0)x).$$

Therefore, we have

$$\beta\geq\left|\frac{1}{2\|x\|^2}z^*\left(\Phi(f,x,x,x)-2\{x,x,x\}\right)\right|, \tag{3.9}$$

where $z^*\in T(\{x,x,x\})$. Further, let

$$h(\zeta)=\frac{\zeta}{2\|x\|^2}z^*(\Psi(f,\zeta,x)),\quad |\zeta|<\frac{1}{\|x\|},$$

where

$$\Psi(f,\zeta,x)$$
$$=[Dg_x(0)]^{-1}[Df(\zeta x)]^{-1}D^2f(\zeta x)(Dg_x(0)x,Dg_x(0)x).$$

Then $h$ is a holomorphic function on $|\zeta|<1/\|x\|$ and by (2.7)

$$\Phi(f,\zeta x,\zeta x,\zeta x)=\zeta^2\Psi(f,\zeta,x),\quad |\zeta|=1.$$

Since $h(0)=0$, for every $r$ with $r<1/\|x\|$, there exists a value of $\zeta$ with $|\zeta|=r$ such that $\mathrm{Re}h(\zeta)\le 0$.

We now replace $x$ by $\zeta x$ and $z^*$ by $\frac{\overline{\zeta}}{|\zeta|}z^*$ in (3.9), where $|\zeta|=1$ so that $\mathrm{Re}h(\zeta)\le 0$. Then we deduce that

$$\beta\ge\left|h(\zeta)-\frac{\|x\|^3}{\|x\|^2}\right|\ge-\Re h(\zeta)+\|x\|\ge\|x\|,$$

because we have used the fact that $\mathrm{Re}h(\zeta)\le 0$. Hence, $\beta\ge\|x\|$ for all $x\in B$. Therefore $\|ord\|_{X,2}\,\mathcal{F}\ge 1$. This completes the proof. □

As a corollary of Proposition 3.6 and Theorem 3.9, we obtain the following result (cf. [23, Theorem 5.1]).

**Corollary 3.10.** *Let B be the unit ball of a JB*-triple X and let $\mathcal{K}(B)$ be the set of normalized convex mappings on B. Then $\|ord\|_{X,2}\,\mathcal{K}(B)=1$.*

Next, we give a result on a lower bound for starlikeness. Hamada and Kohr [11] (cf. [12]) proved the following sufficient condition for starlikeness on the unit ball of a complex Banach space.

**Proposition 3.11.** *Let f be a locally biholomorphic mapping on the unit ball B of a complex Banach space with $f(0)=0$. If*

$$\|[Df(x)]^{-1}D^2f(x)(x,\cdot)\|\le 1,\quad x\in B,$$

*then f is a starlike mapping of order 1/2 on B.*

Using the above sufficient condition, we will prove the following theorem (cf. [23, Theorems 5.5 and 5.7]).

**Theorem 3.12.** *Let $\mathcal{F}$ be a linear-invariant family on the unit ball B of a JB*-triple X with $\|ord\|_{X,1}\,\mathcal{F}=\alpha<\infty$. If $f\in\mathcal{F}$, then f is a starlike mapping of order 1/2 on $B_{r_s}$, where $r_s\in(0,1)$ is the unique solution of the equation*

$$\frac{2r^2+2\alpha r}{(1-r^2)^2}=1.$$

*Proof.* From Lemma 3.8,

$$\|[Df(x)]^{-1}D^2f(x)(Dg_x(0)y,Dg_x(0)z)\|$$
$$\leqq 2\|Dg_x(0)\{y,x,z\}\|+2\alpha\|Dg_x(0)\|\cdot\|y\|\cdot\|z\|.$$

Also, we have $\|Dg_x(0)\|\le 1$ and $\|[Dg_x(0)]^{-1}\|\le 1/(1-\|x\|^2)$ from (2.5) and (2.6). Therefore, putting $y=[Dg_x(0)]^{-1}x$ and $z=[Dg_x(0)]^{-1}w$ with $\|w\|=1$ and using (2.1), we obtain that

$$\|[Df(x)]^{-1}D^2f(x)(x,w)\|$$
$$\leqq 2\|Dg_x(0)\|\cdot\|[Dg_x(0)]^{-1}x\|\cdot\|x\|\cdot\|[Dg_x(0)]^{-1}w\|$$
$$+2\alpha\|Dg_x(0)\|\cdot\|[Dg_x(0)]^{-1}x\|\cdot\|[Dg_x(0)]^{-1}w\|$$
$$\leqq\frac{2r^2+2\alpha r}{(1-r^2)^2},$$

where $r=\|x\|$. From Proposition 3.11, $f$ is a starlike mapping of order 1/2 on $B_{r_s}$. This completes the proof. □

Before to give the following result, we have to introduce some notations, as follows. This result relates the radius of univalence of a linear invariant family with the radius of nonvanishing of this family.

Let

$$r_0=r_0(\mathcal{F})$$
$$=\sup\{r>0: f(x)\ne 0, 0<\|x\|<r, f\in\mathcal{F}\}$$

and let $r_1=r_1(\mathcal{F})$ denote the radius of univalence of the linear invariant family $F$, i.e.

$$r_1=\sup\{r>0: f \text{ is univalent on } B_r, f\in\mathcal{F}\}.$$

Then, we obtain the following result. This result is a generalization of [24, Lemma 2.4], [23, Theorem 5.11], [9, Theorem 3.4] and [10, Theorem 3.5] to the unit ball of a JB*-triple. We remark that if $\|ord\|_{X,1}\,\mathcal{F}=\alpha<\infty$, then $r_0>0$ from Theorem 3.12.

**Theorem 3.13.** *Let $\mathcal{F}$ be a linear invariant family on the unit ball B of a JB*-triple X. Assume that $r_0(\mathcal{F})>0$. Then*

$$r_1=\frac{r_0}{1+\sqrt{1-r_0^2}}.$$

*Proof.* Let $f\in F$ and $r\le\frac{r_0}{1+\sqrt{1-r_0^2}}$. Also, let $y,z\in B_r$ with $y\ne z$. Let

$$F(w;x)$$
$$=[Dg_x(0)]^{-1}[Df(g_x(0))]^{-1}(f(g_x(w))-f(g_x(0))),\tag{3.10}$$

$w,x\in B$, where $g_x$ is the biholomorphic automorphism of $B$, given in (2.3). Clearly, $F(\cdot;x) \in \mathcal{F}$, for all $x \in B$, and if we set $x = y$ and $w = g_y^{-1}(z)$ in (3.10), we obtain that

$$F\left(g_y^{-1}(z);y\right)$$
$$= [Dg_y(0)]^{-1}[Df(y)]^{-1}(f(z)-f(y)). \tag{3.11}$$

From (2.11), we obtain

$$1-\left\|g_{-y}(z)\right\|^2 \geq \frac{(1-\|y\|^2)(1-\|z\|^2)}{(1+\|y\|\cdot\|z\|)^2} > \frac{(1-r^2)^2}{(1+r^2)^2}.$$

Therefore, we have

$$\left\|g_y^{-1}(z)\right\| = \left\|g_{-y}(z)\right\| < \frac{2r}{1+r^2} \leq r_0.$$

Since $g_y^{-1}(z) \neq 0$ for $y \neq z$, we have $F\left(g_y^{-1}(z);y\right) \neq 0$. Then, we conclude from (3.11) that $f(y) \neq f(z)$, that means $f$ is univalent on $B_r$. Therefore, $r_1 \geq \dfrac{r_0}{1+\sqrt{1-r_0^2}}$. Also, since $r_0 > 0$, we deduce that $r_1 > 0$.

In the second part of this proof, we will show that $r_1 \leq \dfrac{r_0}{1+\sqrt{1-r_0^2}}$. To this end, let $x \in B$ with $0 < \|x\| < \dfrac{2r_1}{1+r_1^2}$. Then there exists $a \in B$ such that $x = g_a(a)$ and $0 < \|a\| < r_1$ by (2.8). After short computations, we obtain the following relations

$$F(a;a) = [Dg_a(0)]^{-1}[Df(a)]^{-1}(f(x)-f(a))$$

and

$$F(-a;a) = -[Dg_a(0)]^{-1}[Df(a)]^{-1}f(a),$$

where $F$ is defined by (3.10). Therefore, we have

$$f(x) = Df(a)Dg_a(0)\left(F(a;a)-F(-a;a)\right).$$

Since $0 < \|a\| < r_1$, $F(a;a) \neq F(-a;a)$. Hence, $f(x) \neq 0$. This implies that $r_0 \geq \dfrac{2r_1}{1+r_1^2}$. This is equivalent to $r_1 \leq \dfrac{r_0}{1+\sqrt{1-r_0^2}}$. This completes the proof. □

**Corollary 3.14.** *Let $\mathcal{F}$ be a linear invariant family on the unit ball $B$ of a JB*-triple $X$. Assume that $r_0(\mathcal{F}) = 1$. Then $\mathcal{F}$ is a family of normalized univalent mappings on $B$.*

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