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Article

On some inequalities for a trigonometric polynomial

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Abstract

We give an alternative proof of the Bernstein inequalities and the Szegö inequalities for trigonometric polynomials or polynomials. Later, we show that the n^{th} polarization constant $c(n,H)$ of the Hilbert space H equals 1.

Key Words: Bernstein's inequality, Szegö inequality, trigonometric polynomial

1 Introduction

Let $T:\mathbb{R}\to\mathbb{C}$ be a trigonometric polynomial $T(\theta)=\sum_{k=0}^{n}(\alpha_k\cos k\theta+\beta_k\sin k\theta)$ of degree n with complex coefficients $\alpha_k,\beta_k\in\mathbb{C}$. Let T' be the derivative of T. Then, the following estimate is called the Bernstein inequality (cf. [1]):

$$|T'(\theta)|\le n\max_{\theta\in[0,2\pi]}|T(\theta)| \qquad \text{for } \theta\in\mathbb{R}.$$

For a trigonometric polynomial $T(\theta)=\sum_{k=0}^{n}(a_k\cos k\theta+b_k\sin k\theta)$ with real coefficients $a_k,b_k\in\mathbb{R}$, the following estimate is called the Szegö inequality (cf. [6]):

$$\left\{T(\theta)\right\}^2+\left\{\frac{1}{n}T'(\theta)\right\}^2\le\left(\max_{\theta\in[0,2\pi]}|T(\theta)|\right)^2 \qquad \text{for all } \theta\in\mathbb{R}.$$

In the present paper, we show the Bernstein inequality and the Szegö inequality for a trigonometric polynomial using Lagrange's interpolation polynomial. By using this result, we show the Bernstein inequalities for a polynomial on a Hilbert space. Moreover, we have that the n^{th} polarization constant $c(n,H)$ of the Hilbert space H equals 1.

2 Preliminaries

Let $P:\mathbb{C}\to\mathbb{C}$ be a polynomial of degree $\le n$. We take distinct points $\zeta_1,\dots,\zeta_{n+1}\in\mathbb{C}$. If there exist $z_j\in\mathbb{C}$ such that $P(z_j)=\zeta_j$ for $j=1,\dots,n+1$, then P is unique with degree n. In fact, the unique polynomial is given by

$$P(z)=\sum_{j=1}^{n+1}\frac{\zeta_j q(z)}{(z-z_j)q'(z_j)}, \tag{2.1}$$

where $q(z)=\prod_{j=1}^{n+1}(z-\zeta_j)=(z-\zeta_1)\times\cdots\times(z-\zeta_{n+1})$. Then the polynomial $P(z)$ is called *Lagrange's interpolation polynomial* and the points $\zeta_1,\dots,\zeta_{n+1}$ are called *the interpolation nodes.*

Let $f:\mathbb{C}\to\mathbb{C}$ be a continuous function with distinct points $f(z_j)$ for $j=1,\dots,n+1$. Then the polynomial

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$$P(z)=\sum_{j=1}^{n+1}\frac{f(z_j)q(z)}{(z-z_j)q'(z_j)}, \tag{2.2}$$

is called *Lagrange's interpolation polynomial for f* with nodes at $z_1,\dots,z_{n+1}$.

Using Lagrange's interpolation polynomial, we obtain the following proposition for trigonometric polynomials.

Proposition 2.1. *Let* $T(\theta)=\sum_{k=0}^{n}(\alpha_k\cos k\theta+\beta_k\sin k\theta)$ *be a trigonometric polynomial of degree n with complex coefficients* $\alpha_k,\beta_k\in\mathbb{C}$. *We put* $\theta_j=\frac{2j-1}{2n}\pi$ *for* $j=1,2,\dots,2n$. *Then*

$$T'(\theta)=\frac{1}{2n}\sum_{j=1}^{2n}T\left(\theta+\theta_j\right)\frac{(-1)^{j+1}}{1-\cos\theta_j}\quad \textit{for all } \theta\in\mathbb{R}. \tag{2.3}$$

Proof. We put $z=e^{i\theta}$ and $F(z)=z^nT(\theta)$. Since $\cos k\theta=\frac{z^{2k}+1}{2z^k}$ and $\sin k\theta=\frac{1-z^{2k}}{2z^k}i$, we have

$$F(z)=\sum_{k=0}^{n}\frac{1}{2}\{\alpha_k(z^{2k}+1)z^{n-k}+i\beta_k(1-z^{2k})z^{n-k}\}.$$

Then $F(z)$ is a polynomial of degree $2n$.

We set $G_\zeta(z)=\frac{F(\zeta z)-F(\zeta)}{z-1}$ for $z\in\mathbb{C}$, $\zeta\in\mathbb{C}$. We may assume $\zeta\neq 0$. Then $G_\zeta(z)$ is a polynomial of degree $(2n-1)$ with respect to z. We set $z_j=e^{i\theta_j}$ for $j=1,2,\dots,2n$. Since Lagrange's interpolation polynomial for $G_\zeta(z)$ with nodes at $z_1,\dots,z_{2n}$ is unique, using (2.2), we have

$$G_\zeta(z)=\sum_{j=1}^{2n}\frac{G_\zeta(z_j)q(z)}{(z-z_j)q'(z_j)}. \tag{2.4}$$

Let $q(z)=\prod_{j=1}^{2n}(z-z_j)=z^{2n}+1$. Then $q'(z_j)=-\frac{2n}{z_j}$. So we obtain

$$G_\zeta(z)=\frac{-1}{2n}\sum_{j=1}^{2n}\frac{F(\zeta z_j)-F(\zeta)}{z_j-1}\frac{z^{2n}+1}{z-z_j}z_j. \tag{2.5}$$

From (2.5) and $G_\zeta(1)=\zeta F'(\zeta)$, we have

$$\zeta F'(\zeta)=\frac{1}{2n}\sum_{j=1}^{2n}F(\zeta z_j)\frac{2z_j}{(1-z_j)^2}-\frac{F(\zeta)}{2n}\sum_{j=1}^{2n}\frac{2z_j}{(1-z_j)^2} \tag{2.6}$$

for $\zeta\in\mathbb{C}\setminus\{0\}$. Especially, in case $T(\theta)=\cos n\theta+i\sin n\theta$, we consider $F(z)=z^{2n}$. Then, from (2.6),

$$2n\zeta^{2n}=\frac{1}{2n}\sum_{j=1}^{2n}\zeta^{2n}z_j^{2n}\frac{2z_j}{(1-z_j)^2}-\frac{\zeta^{2n}}{2n}\sum_{j=1}^{2n}\frac{2z_j}{(1-z_j)^2}.$$

Since $\frac{2z_j}{(1-z_j)^2}=\frac{2e^{i\theta_j}}{(1-e^{i\theta_j})^2}=\frac{2}{e^{i\theta_j}-2+e^{-i\theta_j}}=\frac{1}{\cos\theta_j-1}$ and $z_j^{2n}=-1$ for every $j=1,\dots,2n$, we have

$$2n\zeta^{2n}=\frac{-1}{n}\zeta^{2n}\sum_{j=1}^{2n}\frac{2z_j}{(1-z_j)^2}=\frac{-1}{n}\zeta^{2n}\sum_{j=1}^{2n}\frac{1}{\cos\theta_j-1}.$$

Hence

$$-2n^2=\sum_{j=1}^{2n}\frac{2z_j}{(1-z_j)^2}=\sum_{j=1}^{2n}\frac{1}{\cos\theta_j-1}. \tag{2.7}$$

From (2.6) and (2.7), we have

$$\zeta F'(\zeta)=\frac{1}{2n}\sum_{j=1}^{2n}F(\zeta z_j)\frac{2z_j}{(1-z_j)^2}+nF(\zeta). \tag{2.8}$$

Therefore

$$T'(\theta)=\left\{\frac{F(e^{i\theta})}{(e^{i\theta})^n}\right\}'=-n\frac{F(e^{i\theta})}{(e^{i\theta})^{n+1}}ie^{i\theta}+\frac{F'(e^{i\theta})}{(e^{i\theta})^n}ie^{i\theta}=\frac{i}{2n(e^{i\theta})^n}\sum_{j=1}^{2n}F(e^{i\theta}z_j)\frac{2z_j}{(1-z_j)^2}.$$

So we have (2.3). since we have for every $j=1,2,\ldots,2n$,

$$F(e^{i\theta}z_j)=F(e^{i\theta+i\theta_j})=(e^{i\theta+i\theta_j})^nT(\theta+\theta_j)=e^{in\theta}e^{i\frac{2j-1}{2}\pi}T(\theta+\theta_j)=e^{in\theta}i(-1)^{j+1}T(\theta+\theta_j)$$

□

3 Inequalities for a trigonometric polynomial

We give the Bernstein inequality for a trigonometric polynomial by Proposition 2.1 without integration.

Theorem 3.1. *For a trigonometric polynomial* $T(\theta)=\sum_{k=0}^{n}(\alpha_k\cos k\theta+\beta_k\sin k\theta)$ *of degree n with complex coefficients* $\alpha_k,\beta_k\in\mathbb{C}$. *If* $\max_{\theta\in[0,2\pi]}|T(\theta)|\le 1$, *then the following estimate holds:*

$$|T'(\theta)|\le n \qquad \text{for } \theta\in\mathbb{R}.$$

Proof. Since the period of $|T(\theta)|$ and $|T'(\theta)|$ is 2π, by Proposition 2.1, we obtain

$$\begin{aligned}|T'(\theta)|\le\max_{\theta\in[0,2\pi]}|T'(\theta)|&=\max_{\theta\in[0,2\pi]}\left|\frac{1}{2n}\sum_{j=1}^{2n}T\left(\theta+\theta_j\right)\frac{(-1)^{j+1}}{1-\cos\theta_j}\right|\\&\le\frac{1}{2n}\max_{\theta\in[0,2\pi]}\sum_{j=1}^{2n}\left|T\left(\theta+\theta_j\right)\right|\frac{1}{1-\cos\theta_j}=\frac{1}{2n}\max_{\theta\in[0,2\pi]}\left|T\left(\theta\right)\right|\sum_{j=1}^{2n}\frac{1}{1-\cos\theta_j},\end{aligned}$$

where $\theta_j=\dfrac{2j-1}{2n}\pi$ for $j=1,2,\ldots,2n$.

Using (2.7), we obtain $|T'(\theta)|\le n$ for $\theta\in\mathbb{R}$. □

For a trigonometric polynomial $T(\theta)=\sum_{j=0}^{n}(a_j\cos j\theta+b_j\sin j\theta)$ of degree n with real coefficients $a_j,b_j\in\mathbb{R}$, we denote $T^{(k)}(\theta)=\dfrac{d^kT}{d\theta^k}(\theta)$. Using Theorem 4.1, we show the following lemma.

Lemma 3.2. *Let T(θ) be a trigonometric polynomial of degree n with real coefficients and* $\max_{\theta\in[0,2\pi]}|T(\theta)|\le 1$. *If there exists a natural number* $k_0\in\mathbb{N}$ *such that*

$$\left\{\frac{1}{n^{k_0}}T^{(k_0)}(\theta)\right\}^2+\left\{\frac{1}{n^{k_0+1}}T^{(k_0+1)}(\theta)\right\}^2\le 1 \quad \text{for all } \theta\in\mathbb{R},$$

then

$$\left\{\frac{1}{n^{k_0-1}}T^{(k_0-1)}(\theta)\right\}^2+\left\{\frac{1}{n^{k_0}}T^{(k_0)}(\theta)\right\}^2\le 1 \quad \text{for all } \theta\in\mathbb{R}.$$

Proof. By Theorem 3.1, we have the estimate

$$\left|\frac{1}{n^k}T^{(k)}(\theta)\right|\le 1 \tag{3.9}$$

for all $k\in\mathbb{N}$ and all $\theta\in\mathbb{R}$. We set $F(\theta)=\left\{\frac{1}{n^{k_0-1}}T^{(k_0-1)}(\theta)\right\}^2+\left\{\frac{1}{n^{k_0}}T^{(k_0)}(\theta)\right\}^2$. Then,

$$F'(\theta)=2T^{(k_0)}(\theta)\left\{\frac{1}{n^{2(k_0-1)}}T^{(k_0-1)}(\theta)+\frac{1}{n^{2k_0}}T^{(k_0+1)}(\theta)\right\}. \tag{3.10}$$

We assume that $F(\theta_0)$ is the maximum value of F on the closed interval $[0,2\pi]$. Then $T'(\theta_0)=0$. It follows from this and (3.10), that we consider the following two cases.

1. In case that $T^{(k_0)}(\theta_0)=0$.

   Since $\dfrac{1}{n^{k_0-1}}|T^{(k_0-1)}(\theta_0)|\le 1$ from (3.9), we have

$$F(\theta)\le\left\{\frac{1}{n^{k_0-1}}T^{(k_0-1)}(\theta_0)\right\}^2+\left\{\frac{1}{n^{k_0}}T^{(k_0)}(\theta_0)\right\}^2$$
$$\le\left\{\frac{1}{n^{k_0-1}}T^{(k_0-1)}(\theta_0)\right\}^2\le 1\quad\text{for all }\theta\in\mathbb{R}.$$

2. In case that $T^{(k_0-1)}(\theta_0)=-\dfrac{1}{n^2}T^{(k_0+1)}(\theta_0)$.

$$F(\theta)\le\left\{\frac{1}{n^{k_0-1}}T^{(k_0-1)}(\theta_0)\right\}^2+\left\{\frac{1}{n^{k_0}}T^{(k_0)}(\theta_0)\right\}^2$$
$$\le\left\{\frac{1}{n^{k_0+1}}T^{(k_0+1)}(\theta_0)\right\}^2+\left\{\frac{1}{n^{k_0}}T^{(k_0)}(\theta_0)\right\}^2\le 1\quad\text{for all }\theta\in\mathbb{R}.$$

This completes the proof. □

Now we give an alternative proof of the Szegö inequality for a trigonometric polynomial.

Theorem 3.3. *Let* $T(\theta)=\sum_{j=0}^{n}(a_j\cos j\theta+b_j\sin j\theta)$ *be a trigonometric polynomial of degree n with real coefficients* $a_j,b_j\in\mathbb{R}$. *If* $\max_{\theta\in[0,2\pi]}|T(\theta)|\le 1$, *then*

$$\{T(\theta)\}^2+\left\{\frac{1}{n}T'(\theta)\right\}^2\le 1\quad\textit{for all }\theta\in\mathbb{R}.\tag{3.11}$$

Proof. We can calculate the derivative of T:

$$\frac{1}{n^k}T^{(k)}(\theta)=\frac{1}{n^k}\sum_{j=1}^{n}j^k\left\{a_j\cos\left(j\theta+\frac{\pi k}{2}\right)+b_j\sin\left(j\theta+\frac{\pi k}{2}\right)\right\}$$
$$=\sum_{j=1}^{n-1}\left(\frac{j}{n}\right)^k\left\{a_j\cos\left(j\theta+\frac{\pi k}{2}\right)+b_j\sin\left(j\theta+\frac{\pi k}{2}\right)\right\}$$
$$+a_n\cos\left(n\theta+\frac{\pi k}{2}\right)+b_n\sin\left(n\theta+\frac{\pi k}{2}\right),\tag{3.12}$$
$$\frac{1}{n^{k+1}}T^{(k+1)}(\theta)=\sum_{j=1}^{n-1}\left(\frac{j}{n}\right)^{k+1}\left\{a_j\cos\left(j\theta+\frac{\pi(k+1)}{2}\right)+b_j\sin\left(j\theta+\frac{\pi(k+1)}{2}\right)\right\}$$
$$-a_n\sin\left(n\theta+\frac{\pi k}{2}\right)+b_n\cos\left(n\theta+\frac{\pi k}{2}\right).$$

Since $\dfrac{j}{n}<1$ for $j=1,\dots,n-1$, for arbitrary $\varepsilon>0$, there exists a natural number $k_0\in\mathbb{N}$ such that

$$\sum_{j=1}^{n-1}\left(\frac{j}{n}\right)^k(|a_j|+|b_j|)<\varepsilon\quad\text{for }k>k_0.\tag{3.13}$$

So we have

$$\left|\sum_{j=1}^{n-1}\left(\frac{j}{n}\right)^k\left\{a_j\cos\left(j\theta+\frac{\pi k}{2}\right)+b_j\sin\left(j\theta+\frac{\pi k}{2}\right)\right\}\right|<\varepsilon\tag{3.14}$$

and

$$\left|\sum_{j=1}^{n-1}\left(\frac{j}{n}\right)^{k+1}\left\{a_j\cos\left(j\theta+\frac{\pi(k+1)}{2}\right)+b_j\sin\left(j\theta+\frac{\pi(k+1)}{2}\right)\right\}\right|<\varepsilon.$$

Therefore

$$\left|\frac{1}{n^k}T^{(k)}(\theta)\right| < \varepsilon + \left|a_n \cos\left(n\theta + \frac{\pi k}{2}\right) + b_n \sin\left(n\theta + \frac{\pi k}{2}\right)\right|,$$
$$\left|\frac{1}{n^{k+1}}T^{(k+1)}(\theta)\right| < \varepsilon + \left|a_n \sin\left(n\theta + \frac{\pi k}{2}\right) - b_n \cos\left(n\theta + \frac{\pi k}{2}\right)\right|$$

By Theorem 3.1, we have for all $k \in \mathbb{N}$ and all $\theta \in \mathbb{R}$

$$\left|\frac{1}{n^k}T^{(k)}(\theta)\right| \le 1. \tag{3.15}$$

From (3.12), we have

$$a_n \cos\left(n\theta + \frac{\pi k}{2}\right) + b_n \sin\left(n\theta + \frac{\pi k}{2}\right)$$
$$= \frac{1}{n^k}T^{(k)}(\theta) - \sum_{j=1}^{n-1}\left(\frac{j}{n}\right)^k \left\{a_j \cos\left(j\theta + \frac{\pi k}{2}\right) + b_j \sin\left(j\theta + \frac{\pi k}{2}\right)\right\}.$$

It follows from (3.14) and (3.15) that

$$\left|a_n \cos\left(n\theta + \frac{\pi k}{2}\right) + b_n \sin\left(n\theta + \frac{\pi k}{2}\right)\right| < 1 + \varepsilon \quad (\theta \in \mathbb{R})$$

Then we have

$$\sqrt{a_n^2 + b_n^2} < 1 + \varepsilon. \tag{3.16}$$

Hence, from (3.13) and (3.16),

$$\left\{\frac{1}{n^k}T^{(k)}(\theta)\right\}^2 + \left\{\frac{1}{n^{k+1}}T^{(k+1)}(\theta)\right\}^2$$
$$< \left\{\varepsilon + \left|a_n \cos\left(n\theta + \frac{\pi k}{2}\right) + b_n \sin\left(n\theta + \frac{\pi k}{2}\right)\right|\right\}^2$$
$$+ \left\{\varepsilon + \left|a_n \sin\left(n\theta + \frac{\pi k}{2}\right) - b_n \cos\left(n\theta + \frac{\pi k}{2}\right)\right|\right\}^2$$
$$= 2\varepsilon^2 + a_n^2 + b_n^2 + 2\varepsilon\left|a_n \cos\left(n\theta + \frac{\pi k}{2}\right) + b_n \sin\left(n\theta + \frac{\pi k}{2}\right)\right|$$
$$+ 2\varepsilon\left|a_n \sin\left(n\theta + \frac{\pi k}{2}\right) - b_n \cos\left(n\theta + \frac{\pi k}{2}\right)\right|$$
$$< 2\varepsilon^2 + (1+\varepsilon)^2 + 4\varepsilon(1+\varepsilon) = 1 + 6\varepsilon + 7\varepsilon^2.$$

Letting ε tend to 0, we obtain

$$\left\{\frac{1}{n^k}T^{(k)}(\theta)\right\}^2 + \left\{\frac{1}{n^{k+1}}T^{(k+1)}(\theta)\right\}^2 \le 1$$

for $k \ge k_0$ and all $\theta \in \mathbb{R}$.

Applying Lemma 3.2 inductively, we have (3.11). This completes the proof.

□

4 Applications of the inequalities

Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $\le n$ on the complex plane $\mathbb{C}$ and P' be the derivative of P. We set $\| P \| = \sup_{|z|\le 1} |P(z)|$ and $\| P' \| = \sup_{|z|\le 1} |P'(z)|$. Then we show the following inequality called the Bernstein inequality for a polynomial (cf. [1]).

Theorem 4.1. *For a polynomial $P:\mathbb{C}\to\mathbb{C}$ of degree n with complex coefficients, the following estimate holds:*

$$\| P' \| \le n \| P \| .$$

Proof. We set $T(\theta)=P(e^{i\theta})$. Then, since $T'(\theta)=P'(e^{i\theta})ie^{i\theta}$, we have $|T'(\theta)|=|P'(e^{i\theta})|$. By Theorem 3.1,

$$\| P' \| = \sup_{|z|=1} |P'(z)| = \sup_{\theta\in[0,2\pi]} |P'(e^{i\theta})| = \max_{\theta\in[0,2\pi]} |T'(\theta)| \le \max_{\theta\in[0,2\pi]} n|T(\theta)| = n\|P\| .$$

□

Let H be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$. We set $\bar{B}_H=\{x\in H;\|x\|_H=\langle x,x\rangle^{\frac{1}{2}}\le 1\}$. Let X be a normed space with the norm $\|\cdot\|_X$, $\check{p}:H^n\to X$ be a continuous n-linear mapping and $p(x)=\check{p}(x,x,\cdots,x)$ be the corresponding homogeneous polynomial of degree n for $n\in\mathbb{N}$. We define by

$$\| p \| = \sup\{\| p(x) \|_X ; x\in \bar{B}_H\},$$
$$\| \check{p} \| = \sup\{\| \check{p}(x_1,\cdots,x_n) \|_X ; x_j\in \bar{B}_H, j=1,\cdots,n\}.$$

Let $Dp(x)$ denote the Fréchet derivative of p at x. Then

$$Dp(x)y=\frac{d}{dt}p(x+ty)\Big|_{t=0}$$

for $y\in H\setminus\{0\}$. In fact, $p(x+ty)=\check{p}(x+ty,\cdots,x+ty)=\check{p}(x,\ldots,x)+tn\check{p}(x,\ldots,x,y)+\sum_{j=2}^{n} t^j\check{p}(\overbrace{x,\ldots,x}^{n-j},\overbrace{y\ldots,y}^{j})$. Hence

$$Dp(x)y=n\check{p}(x,\ldots,x,y). \tag{4.17}$$

We denote by $\mathcal{P}({}^nH;X)$ the space of all homogeneous polynomial of degree n. When $X=\mathbb{C}$, we write $\mathcal{P}({}^nH)$ in place of $\mathcal{P}({}^nH;X)$.

Proposition 4.2. *Let H be a complex Hilbert space and X be a complex normed space. For $p\in\mathcal{P}({}^nH;X)$,*

$$\| Dp \| \le n \| p \| .$$

Proof. We put

$$\sigma=\begin{cases}1 & \text{if } \langle x,y\rangle=0,\\ \dfrac{\langle x,y\rangle}{|\langle x,y\rangle|} & \text{if } \langle x,y\rangle\ne 0\end{cases}$$

for $x,y\in H$. Since

$$\begin{aligned}\| x\cos\theta+i\sigma y\sin\theta\|_H^2 &= \langle x\cos\theta+i\sigma y\sin\theta, x\cos\theta+i\sigma y\sin\theta\rangle\\ &\le \max\{\|x\|_H^2,\|y\|_H^2\}\end{aligned}$$

for $x,y\in\bar{B}_H$, we have $x\cos\theta+i\sigma y\sin\theta\in\bar{B}_H$. Then

$$x\cos\theta+i\sigma y\sin\theta = x\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\theta^{2k}+i\sigma y\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\theta^{2k+1}=x+i\sigma y\theta+\sum_{k\ge 2}a_k\theta^k,$$

where a_k are some complex numbers.

We take a linear functional $\varphi:X\to\mathbb{C}$ with $\|\varphi\|=1$ and put $T_n(\theta)=\varphi\circ p(x\cos\theta+i\sigma y\sin\theta)$. Then $T_n(\theta)$ is a trigonometric polynomial of degree n with complex coefficients and

$$\begin{aligned}T_n(\theta) &= \varphi\circ p(x+i\sigma y\theta+\sum_{k\ge 2}a_k\theta^k)\\ &= \varphi\circ\check{p}(x+i\sigma y\theta+\sum_{k\ge 2}a_k\theta^k,\cdots,x+i\sigma y\theta+\sum_{k\ge 2}a_k\theta^k)\\ &= \varphi\circ\check{p}(x,\cdots,x)+n\theta\varphi\circ\check{p}(x,\cdots,x,i\sigma y)+\sum_{k\ge 2}b_k\theta^k,\end{aligned}$$

where b_k are some complex numbers. Hence

$$T_n'(0) = \frac{d}{d\theta} T_n(\theta)\Big|_{\theta=0} = i\sigma n\varphi \circ \check{p}(x, \cdots, x, y). \tag{4.18}$$

By Theorem 3.1, (4.17) and (4.18),

$$\varphi\big(Dp(x)y\big)| = n\,|\,\varphi \circ \check{p}(x, \cdots, x, y)\,| = |\,T_n'(0)\,| \le n \max_{\theta\in[0,2\pi]} |\,T_n(\theta)\,| \le n \,\|\, p \,\|$$

By the Hahn-Banach theorem, we can choose φ satisfying $\varphi\big(Dp(x)y\big) = \| Dp(x)y \|_X$. Therefore

$$\| Dp(x)y \|_X \le n \,\| p \| \tag{4.19}$$

for $x, y \in \bar{B}_H$. □

For $p \in \mathcal{P}(^n H)$, we set $c(n,H) = \inf\{M; \| \check{p} \| \le M \| p \| \text{ for all } p \in \mathcal{P}(^n H)\}$. This $c(n, H)$ is called the n^{th} polarization constant of the Hilbert space H and $c(H) = \lim_{n\to\infty} \sup c(n,H)^{\frac{1}{n}}$ is called the polarization constant of the space H. It is clear that $c(n,H) \ge 1$, since $\| p \| \le \| \check{p} \|$.

Using Proposition 4.2, we have the Bernstein inequality for a polynomial and the polarization constant of the complex Hilbert space H

Theorem 4.3. *Let H be a complex Hilbert space and X be a complex normed space. For $p \in \mathcal{P}(^n H; X)$,*

$$\| \check{p} \| = \| p \|.$$

Proof. As in the proof of Proposition 4.2, it follows from (4.7) and (4.19) that

$$\| \check{p}(x, \cdots, x, y_n) \|_X \le \| p \|$$

for $x, y_n \in \overline{B_H}$. We fix y_n and put $p_{n-1}(x) = \check{p}(x, \cdots, x, y_n)$. Then we can look upon $p_{n-1}(x)$ as homogeneous polynomial of degree $(n-1)$ and

$$\| p_{n-1} \| = \sup\{\| \check{p}_{n-1}(x, \cdots, x, y_n) \|_X ; x \in \bar{B}_H\} \le \| p \|.$$

As in the above, we get $\| \check{p}(x, \cdots, x, y_{n-1}, y_n) \|_X \le \| p_{n-1} \|$. It follows from the above analogue by induction that we have

$$\| \check{p}(x, y_2, \cdots, y_{n-1}, y_n) \|_X \le \| p \|.$$

Hence

$$\| \check{p} \| \le \| p \|$$

□

Corollary 4.4. *If H is a complex Hilbert space, then $c(n, H) = c(H) = 1$.*

Using the Szegö inequality for trigonometric polynomials with real coefficients, we obtain we have the Bernstein inequality for a polynomial and the polarization constant of the real Hilbert space H

Proposition 4.5. *Let H be a real Hilbert space and X be a real normed space. For $p \in \mathcal{P}(^n H; X)$,*

$$\| Dp \| \le n \,\| p \|.$$

Proof. We may assume that $\| p \| = 1$ without loss of generality.

We put $w = \dfrac{y - x\langle x, y\rangle}{\| y - x\langle x, y\rangle \|_H}$ for $x, y \in \bar{B}_H$, $x \ne y$. Then $\| x\cos\theta + w\sin\theta \|_H^2 \le 1$. We take a linear functional $\varphi : X \to \mathbb{R}$ with $\| \varphi \| = 1$ and put $T_n(\theta) = \varphi \circ p(x\cos\theta + w\sin\theta)$. Then $T_n(0) = \varphi \circ p(x)$.

As in the proof of Proposition 4.2, $T_n'(0) = \varphi\big(Dp(x)w\big)$. From (4.17),

$$\varphi\big(Dp(x)x\big) = n\varphi \circ p(x) = nT_n(0).$$

Therefore, by Theorem 3.3,

$$\begin{aligned}\varphi\big(Dp(x)y\big) &= \varphi\big(Dp(x)\big(x\langle x, y\rangle + w \parallel y - x\langle x, y\rangle \parallel_H\big)\big)\\ &= \langle x, y\rangle\varphi\big(Dp(x)x\big) + \parallel y - x\langle x, y\rangle \parallel_H \varphi\big(Dp(x)w\big)\\ &= \langle x, y\rangle nT_n(0) + \parallel y - x\langle x, y\rangle \parallel_H T_n'(0)\\ &\leq \sqrt{\langle x, y\rangle^2 + \parallel y - x\langle x, y\rangle \parallel_H^2}\sqrt{n^2T_n(0)^2 + T_n'(0)^2}\\ &\leq n.\end{aligned}$$

By the Hahn-Banach theorem, we can choose φ satisfying $\varphi\big(Dp(x)y\big) = \parallel Dp(x)y \parallel_X$. Therefore $\parallel Dp(x)y \parallel_X \leq n \parallel p \parallel$ for $x, y \in \bar{B}_H$. □

Theorem 4.6. *Let H be a real Hilbert space and X be a real normed space. For $p \in \mathcal{P}(^n H; X)$,*

$$\parallel \check{p} \parallel = \parallel p \parallel.$$

Proof. As in the proof of Theorem 4.3, using inductively Proposition 4.5, we show this theorem. □

Corollary 4.7. *If H is a real Hilbert space, then $c(n, H) = c(H) = 1$.*

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