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論　　文

Bohr's Inequality on the Unit Ball of J*-algebra

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Abstract

Let G be a bounded balanced domain in a complex Banach space X and B_Y be the unit ball in a J*-algebra Y. We will generalise Bohr's theorem to holomorphic mappings f from G into B_Y.

Key Words: Bohr's inequality, J*-algebra

1 Introduction

We first recall Bohr's theorem for the open unit disc Δ in the complex plane $\mathbb{C}$. Let $\Delta = \{z \in \mathbb{C}; |z| < 1\}$ be the open unit disc in $\mathbb{C}$, and let $f : \Delta \to \Delta$ be a holomorphic function with Taylor expansion
$f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then the following inequality holds :

$$\sum_{k=0}^{\infty} |a_k z^k| < 1 \quad \text{for } |z| < \frac{1}{3}.$$

This result was originally obtained in Bohr [8] for $|z| < 1/6$. The fact that the inequality is actually true for $|z| < 1/3$ and Riesz, Schur and Wiener independently showed that the constant $1/3$ is best possible. Other proofs were given by [22] and [23].

It is natural to consider an extension of the above result to more general domains or higher dimensional spaces. Recently, many mathematician obtained multidimensional generalisations of Bohr's theorem (cf. [1], [2], [6], [7], [12], [13]). Such generalisations were obtained by studying the power series of a holomorphic function defined in bounded complete Reinhardt domains in $\mathbb{C}^n$. These results can be summarized as

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follows:

$$\frac{1}{3\sqrt[3]{e}} < K \leq \frac{1}{3} \qquad \text{if } 0 < p \leq 1, \tag{1.1}$$

$$\frac{1}{3\sqrt[3]{e}}\frac{1}{n^{1-1/p}} \leq K < 3\left(\frac{\log n}{n}\right)^{1-1/p} \qquad \text{if } 1 \leq p \leq 2, \tag{1.2}$$

$$\frac{1}{3}\frac{1}{\sqrt{n}} \leq K < 2\sqrt{\frac{\log n}{n}} \qquad \text{if } 2 \leq p \leq \infty, \tag{1.3}$$

where K is the supremum of $r \in [0,1]$ such that $\sum_{\alpha \geq 0} |c_\alpha z^\alpha| < 1$ for $z \in rB_{\ell_p^n}$ whenever $|\sum_{\alpha \geq 0} c_\alpha z^\alpha| < 1$ for $z \in B_{\ell_p^n}$. Here, the sum is taken over multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, such that α_j are nonnegative integers,

$$B_{\ell_p^n} = \left\{ z \in \mathbb{C}^n : \|z\|_p = \left(\sum_{k=1}^n |z_k|^p \right)^{1/p} < 1 \right\}.$$

However, the above results do not give a complete generalisation of Bohr's theorem to several complex variables. Also, if $p > 1$, the above results cannot be generalised to infinite dimensional spaces. These can be verified by putting $n = 1$ or letting $n \to \infty$ in the equations (1.2) and (1.3).

The aim of this paper is to prove the following theorem.

Main Theorem. *Let X be a complex Banach space, Y be a J^*-algebra. Let G be a bounded balanced domain in X and let B_Y be the unit ball in Y. Let $f : G \to B_Y$ be a holomorphic mapping. If $P = f(0)$, then we have*

$$\sum_{k=0}^{\infty} \frac{\|D\varphi_P(P)[D^k f(0)(z^k)]\|}{k!\|D\varphi_P(P)\|} < 1 \tag{1.4}$$

for $z \in (1/3)G$, where $\varphi_P \in \mathrm{Aut}(B_Y)$ such that $\varphi_P(P) = 0$. Moreover, the constant $1/3$ is best possible.

Our result also generalises the above result due to Liu and Wang [18]. Our proof is more simple than that of Liu and Wang [18].

2 Preliminaries

Let X, Y be complex Banach spaces and let B_Y be the unit ball in Y. For domains $G \subset X$, $D \subset Y$, we denote by $H(G, D)$ the set of all holomorphic mappings from G into D. For $f \in H(G, D)$ and $x \in G$, let $D^k f(x)$ denote the k-th Fréchet derivative of f at x. Any mapping $f \in H(G, D)$ can be expanded into the series

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k f(0)(x^k)$$

in a neighbourhood of the origin.

For any $P \in G$, $\xi \in X$,

$$\gamma_G(P,\xi) = \sup\{|Dg(P)\xi| \,:\, g \in H(G,\Delta), g(P) = 0\}$$

is called the infinitesimal Carathéodory pseudometric on G, where Δ is the unit disc in $\mathbb{C}$. Also,

$$K_G(P,\xi) = \inf\left\{\frac{1}{\alpha} \,:\, h \in H(\Delta, G), h(0) = P, Dh(0) = \alpha\xi\right\}$$

is called the infinitesimal Kobayashi pseudometric on G.

A domain $G \subset X$ is said to be balanced, if $zG \subset G$ for all $z \in \overline{\Delta}$. The Minkowski function h of G is defined by

$$h(x) = \inf\{t > 0 : \, x/t \in G\}$$

for $x \in X$. Then we have $G = \{x \in X : \, h(x) < 1\}$.

A mapping $f \in H(G,Y)$ is said to be biholomorphic if $f(G)$ is a domain, the inverse f^{-1} exists and is holomorphic on $f(G)$. We denote by $\mathrm{Aut}(G)$ the set of all biholomorphic mappings of G onto itself.

Let $L(X,Y)$ denote the set of continuous linear operators from X into Y. Let I be the identity in $L(X,X)$. For each $x \in X \setminus \{0\}$, we set

$$T(x) = \{l_x \in L(X,\mathbb{C}) : \; l_x(x) = \|x\|, \; \|l_x\| = 1\}.$$

Then $T(x)$ is nonempty by the Hahn-Banach theorem.

Let H and K be complex Hilbert spaces. For each operator $A \in L(H,K)$, there exists a uniquely determined operator $A^* \in L(K,H)$ such that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x \in H$ and $y \in K$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in a complex Hilbert space. A closed complex linear subspace L of $L(H,K)$ is called a J*-algebra, if $AA^*A \in L$ for all $A \in L$. Harris [15, Theorem 2] gave the following explicit formula for Möbius transformations of the unit ball of a J*-algebra.

Proposition 2.1 *Let B be the unit ball of a J*-algebra X. Then, for each $P \in B$, the Möbius transformation*

$$T_P(Q) = (I - PP^*)^{-1/2}(Q + P)(I + P^*Q)^{-1}(I - P^*P)^{1/2}$$

is a biholomorphic mapping of B onto itself with $T_P(0) = P$. Moreover,

$$T_P^{-1} = T_{-P}, T_P(Q)^* = T_{P^*}(Q^*), \|T_P(Q)\| \leq T_{\|P\|}(\|Q\|)$$

and

$$DT_P(Q)R = (I - PP^*)^{1/2}(I + QP^*)^{-1}R(I + P^*Q)^{-1}(I - P^*P)^{1/2}$$

for $Q \in B$ *and* $R \in X$.

3 Bohr's theorem

To prove Main Theorem, we need the following lemma.

LEMMA 3.1 *Let* B *be the unit ball of a* J^**-algebra. Then, for any* $P \in B$*, there exists a* $\varphi_P \in \mathrm{Aut}(B)$ *such that* $\varphi_P(P) = 0$ *and*

$$\|D\varphi_P(P)\| = \frac{1}{1 - \|P\|^2}.$$

Proof: Let

$$\varphi_P(Q) = (I - PP^*)^{-1/2}(Q - P)(I - P^*Q)^{-1}(I - P^*P)^{1/2}.$$

Then, by Proposition 2.1, φ_P is an automorphism of B such that $\varphi_P(P) = 0$ and

$$D\varphi_P(P)Q = (I - PP^*)^{-1/2}Q(I - P^*P)^{-1/2}.$$

If $P = 0$, then we have $D\varphi_P(P)Q = Q$. So, $\|D\varphi_P(P)\| = 1 = \dfrac{1}{1 - \|P\|^2}$.

We will consider the case $P \neq 0$. We set

$$g(Q) = \frac{l_P(Q) - \|P\|}{1 - \|P\| l_P(Q)}$$

where $l_P \in T(P)$. Then $g : B \longrightarrow \Delta$ is holomorphic. Since $l_P(\zeta P) = \zeta l_P(P) = \zeta\|P\|$, we have

$$g(\zeta P) = \frac{(\zeta - 1)\|P\|}{1 - \zeta\|P\|^2} \quad \text{for } \zeta \in \overline{\Delta}.$$

Therefore,

$$Dg(\zeta P)P = \frac{\|P\|(1 - \|P\|^2)}{(1 - \zeta\|P\|^2)^2} \quad \text{for } \zeta \in \overline{\Delta}.$$

Putting $\zeta = 1$,

$$Dg(P)P = \frac{\|P\|}{1 - \|P\|^2}.$$

Since $g \circ \varphi_P^{-1} \in H(B, \Delta)$ and $g \circ \varphi_P^{-1}(0) = 0$, using the infinitesimal Carathéodory pseudometric γ_B on the unit ball B, we have

$$\|D\varphi_P(P)P\| = \gamma_B(0, D\varphi_P(P)P) \geq |Dg(P)[D\varphi_P(P)]^{-1}D\varphi_P(P)P| = \frac{\|P\|}{1 - \|P\|^2}.$$

On the other hand, we set

$$h(\zeta) = \frac{\zeta + \|P\|}{1 + \|P\|\zeta} \frac{P}{\|P\|}.$$

Then $h : \Delta \longrightarrow B$ is holomorphic. Since $Dh(\zeta) = \dfrac{P(1 - \|P\|^2)}{\|P\|(1 + \zeta\|P\|)^2}$, we have

$$Dh(0) = \frac{1 - \|P\|^2}{\|P\|} P.$$

Since $\varphi_P \circ h \in H(\Delta, B)$, $\varphi_P \circ h(0) = 0$ and $D(\varphi_P \circ h)(0) = \dfrac{1 - \|P\|^2}{\|P\|} D\varphi_P(P)P$, using the infinitesimal Carathéodory pseudometric K_B on B, we have

$$\|D\varphi_P(P)P\| \;=\; K_B(0, D\varphi_P(P)P) \leq \frac{\|P\|}{1 - \|P\|^2}.$$

Therefore,

$$\|D\varphi_P(P)P\| \;=\; \frac{\|P\|}{1 - \|P\|^2}.$$

Since $\|P^*x\|^2 = \langle P^*x, P^*x\rangle = \langle PP^*x, x\rangle \leq \|P\|\|P^*x\|\|x\|$, we have $\|P^*\| \leq \|P\|$. Therefore,

$$\begin{aligned} \|D\varphi_P(P)Q\| &\leq (1 - \|P\|^2)^{-1/2}\|Q\|(1 - \|P\|^2)^{-1/2} \\ &= \frac{\|Q\|}{1 - \|P\|^2}. \end{aligned}$$

Thus,

$$\begin{aligned} \|D\varphi_P(P)\| &= \sup_{Q \in \overline{B}} \|D\varphi_P(P)Q\| \\ &\leq \sup_{Q \in \overline{B}} \frac{\|Q\|}{1 - \|P\|^2} \\ &= \frac{1}{1 - \|P\|^2} \\ &= \frac{1}{\|P\|} \frac{\|P\|}{1 - \|P\|^2} \\ &= \frac{1}{\|P\|} \|D\varphi_P(P)P\| \\ &= \|D\varphi_P(P) \frac{P}{\|P\|}\| \\ &\leq \|D\varphi_P(P)\|. \end{aligned}$$

This completes the proof. ■

Theorem 3.2 (Main Theorem) *Let X be a complex Banach space, Y be a J^*-algebra. Let G be a bounded balanced domain in X and let B_Y be the unit ball in Y. Let $f : G \to B_Y$ be a holomorphic mapping. If $P = f(0)$, then we have*

$$\sum_{k=0}^{\infty} \frac{\|D\varphi_P(P)[D^k f(0)(z^k)]\|}{k!\|D\varphi_P(P)\|} < 1 \tag{3.1}$$

for $z \in (1/3)G$, where $\varphi_P \in \mathrm{Aut}(B_Y)$ such that $\varphi_P(P) = 0$. Moreover, the constant $1/3$ is best possible.

Proof: For a fixed positive integer k, we set

$$f_k(z) = \sum_{j=1}^{k} \frac{f(e^{i2\pi j/k} z)}{k}.$$

Since G is balanced and B_Y is convex, $f_k \in H(G, B_Y)$. From the homogeneous expansion $f(z) = f(0) + \sum_{l=1}^{\infty} \frac{D^l f(0)(z^l)}{l!}$, we have

$$f_k(z) = \frac{1}{k}\sum_{j=1}^{k}\left(f(0) + \sum_{l=1}^{\infty} \frac{D^l f(0)\left((e^{i2\pi j/k}z)^l\right)}{l!}\right) = \frac{1}{k}\sum_{j=1}^{k}\left(P + \sum_{l=1}^{\infty} e^{i2\pi jl/k}\frac{D^l f(0)(z^l)}{l!}\right)$$

for z sufficiently close to the origin. Since

$$\frac{1}{k}\sum_{j=1}^{k} e^{i2\pi jl/k} = \begin{cases} 1 & \text{if } l \equiv 0 \pmod{k}, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$f_k(z) = \frac{1}{k}\sum_{j=1}^{k} P + \frac{1}{k}\sum_{j=1}^{k}\sum_{l=1}^{\infty} e^{i2\pi jl/k}\frac{D^l f(0)(z^l)}{l!} = P + \sum_{m=1}^{\infty}\frac{D^{km} f(0)(z^{km})}{(km)!}.$$

From the Taylor expansion of φ_P at P, we have

$$\begin{aligned} \varphi_P \circ f_k(z) &= \varphi_P\left(P + \sum_{m=1}^{\infty}\frac{D^{km} f(0)(z^{km})}{(km)!}\right) \\ &= \varphi_P(P) + D\varphi_P(P)\left(P + \sum_{m=1}^{\infty}\frac{D^{km} f(0)(z^{km})}{(km)!} - P\right) + \cdots \\ &= \frac{D\varphi_P(P)[D^k f(0)(z^k)]}{k!} + \frac{D\varphi_P(P)[D^{2k} f(0)(z^{2k})]}{(2k)!} + \cdots \end{aligned}$$

for z sufficiently close to the origin. Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_P \circ f_k(e^{i\theta}z)e^{-ik\theta}d\theta = \frac{D\varphi_P(P)[D^k f(0)(z^k)]}{k!} \tag{3.2}$$

for z sufficiently close to the origin. By the identity theorem for holomorphic mappings, this equality (3.2) holds for $z \in G$. From this equality and the fact that $\varphi_P \circ f_k \in H(G, B_Y)$, we have

$$\frac{\|D\varphi_P(P)[D^k f(0)(z^k)]\|}{k!} \le \frac{1}{2\pi}\int_0^{2\pi} \|\varphi_P \circ f_k(e^{i\theta}z)\| \left|e^{-ik\theta}\right| d\theta < 1 \quad \text{for } z \in G.$$

By the Schwarz lemma for holomorphic mappings on bounded balanced domain,

$$\frac{\|D\varphi_P(P)[D^k f(0)(z^k)]\|}{k!} \le h(z)^k \quad \text{for } z \in \overline{G},$$

where h is the Minkowski function of G. By Lemma 3.1, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{\|D\varphi_P(P)[D^k f(0)(z^k)]\|}{k!\|D\varphi_P(P)\|} &= \frac{\|D\varphi_P(P)[f(0)]\|}{0!\,\|D\varphi_P(P)\|} \\
&\quad + \sum_{k=1}^{\infty} \frac{\|D\varphi_P(P)[D^k f(0)(z^k)]\|}{k!} \frac{1}{\|D\varphi_P(P)\|} \\
&\le \|P\| + \sum_{k=1}^{\infty} h(z)^k (1 - \|P\|^2) \\
&= \|P\| + \frac{h(z)}{1-h(z)}(1-\|P\|)(1+\|P\|).
\end{aligned}$$

Thus, if $h(z) < 1/3$, then

$$\sum_{k=0}^{\infty} \frac{\|D\varphi_P(P)[D^k f(0)(z^k)]\|}{k!\|D\varphi_P(P)\|} < \|P\| + \frac{1/3}{1-1/3}(1-\|P\|)(1+1) = 1.$$

Finally, we will show that the constant $1/3$ is best possible.

For any $r \in (1/3, 1)$, there exists a $c \in (0,1)$ such that $cr > 1/3$. Then there exist $V \in \partial G$ and $\lambda \in (0,1)$ such that

$$c \sup\{\|x\| : x \in \partial G\} < \|V\| \text{ and } cr > \frac{1}{1+2\lambda}.$$

For any $U \in \partial B_Y$, we set

$$P = \lambda U \text{ and } F(\zeta) = \frac{\lambda - \zeta}{1 - \lambda\zeta}.$$

Let

$$f(z) = F\left(c\frac{l_V(z)}{\|V\|}\right) U,$$

where $l_V \in T(V)$. Then $f : G \longrightarrow B_Y$ is holomorphic and

$$\begin{aligned}
D^k f(0)(z^k) &= D^k F(0)\left(c\frac{l_V(z)}{\|V\|}\right)^k U \\
&= k!(\lambda^{k+1} - \lambda^{k-1})(cr)^k U \\
&= k!(\lambda^2 - 1)\lambda^{k-1}(cr)^k \frac{1}{\lambda} P.
\end{aligned}$$

Let $\varphi_P \in \mathrm{Aut}(B_Y)$ be as in Lemma 3.1. Then, for $z = rV$, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{\|D\varphi_P(P)[D^k f(0)(z^k)]\|}{k!\|D\varphi_P(P)\|} &= \frac{\|D\varphi_P(P)[f(0)]\|}{0!\,\|D\varphi_P(P)\|} + \sum_{k=1}^{\infty} \frac{\|D\varphi_P(P)[D^k f(0)(z^k)]\|}{k!\|D\varphi_P(P)\|} \\
&= \|P\| + \sum_{k=1}^{\infty} |\lambda^2 - 1|\lambda^{k-1}(cr)^k \frac{1}{\lambda} \frac{\|D\varphi_P(P)P]\|}{\|D\varphi_P(P)\|} \\
&= \lambda + (1-\lambda^2)cr \sum_{k=1}^{\infty} (\lambda cr)^{k-1} \\
&= \lambda + \frac{(1-\lambda^2)(cr)}{1-\lambda cr} \\
&> \lambda + \frac{(1-\lambda^2)\dfrac{1}{1+2\lambda}}{1-\lambda\dfrac{1}{1+2\lambda}} \\
&= 1.
\end{aligned}$$

This implies that the constant 1/3 is best possible. This completes the proof. ■

As a corollary, we obtain Bohr's theorem for holomorphic mappings on bounded balanced domains of a complex Banach space with values in the unit disc Δ in $\mathbb{C}$.

COROLLARY 3.3 *Let X be a complex Banach space and let G be a bounded balanced domain in X. Let $f : G \to \Delta$ be a holomorphic mapping. Then we have*

$$\sum_{k=0}^{\infty} \frac{|D^k f(0)(z^k)|}{k!} < 1$$

for $z \in (1/3)G$. Moreover, the constant $1/3$ is best possible.

Remark 3.4 Let B_Y be one of the four classical domains in the sense of [16]. Then B_Y is the unit ball of a J*-algebra [15]. Hence, the above theorem generalises a result due to [18]. On the other hand, in [1, Theorem 8], he obtained the above corollary when G is a bounded balanced domain in $\mathbb{C}^n$. However, in [1], he assumed that G is convex to deduce that the constant 1/3 is best possible. In the above corollary, we do not need the convexity of G.

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